1. If \( f(x) = \left( \frac{x}{1 + \frac{1}{x}} \right)^{-1} \), what is \( f\left(\frac{3}{2}\right) \)?

\[ f\left(\frac{3}{2}\right) = \frac{3}{2} + 1 = \frac{5}{2} \]

2. Find all \( x \) and only those \( x \) that satisfy \( \sqrt{x + 2} = 2x + 1 \).

\[ \sqrt{x + 2} = 2x + 1 \]

3. Write \( \left[(a+b)^3 - (a-b)^3\right] - \left[(a+b)^3 + (a-b)^3\right]^2 \) in the form \( k[f(a,b)]^3 \) where \( k \) is an integer and \( f(a,b) \) is an expression of two terms involving \( a \) and \( b \).

\[ -4 \left[ a^2 - b^2 \right]^3 = 4 \left[ b^2 - a^2 \right]^3 \]

4. Express \( \sum_{n=1}^{100} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{10100} \) as the quotient of two relatively prime integers.

\[ \frac{100}{101} \]

1. \( f(x) = \left( \frac{x^2}{x + 1} \right)^{-1} \)

\[ f\left(\frac{3}{2}\right) = \frac{3^2 + 1}{\sqrt{4}} = \frac{10}{2} = 5 \]

2. \( x + 2 = 4x^2 + 4x + 1 \)

\[ 4x^2 + 3x - 1 = 0 \]

\[ (4x - 1)(x + 1) = 0 \]

It is necessary to check the answers. \( x = -1 \) does not check, \( x = \frac{1}{4} \)

3. \( \left[(a+b)^3 - (a-b)^3\right] - \left[(a+b)^3 + (a-b)^3\right]^2 \)

\[ = -4 \left[(a+b)^3 - (a-b)^3\right] = -4 \left[(a+b)(a+b)^2\right]^3 = -4 \left[a^2 - b^2\right]^3 \]

4. \( \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \)

The series may then be written

\[ \sum_{n=1}^{100} \left(\frac{1}{n} - \frac{1}{n+1}\right) = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{100} - \frac{1}{101}) \]

\[ = 1 - \frac{1}{101} = \frac{100}{101} \]
1. Figure 1 shows a regular octagon $ABCDEFGH$ inscribed in a circle of radius 1. Lines tangent to the circle at $A$ and $D$ intersect at $R$. What is the measure in degrees of $\angle ARD$?

2. Referring to the same inscribed octagon described in problem 1, a line tangent to the circle at $C$ intersects the extension of $AG$ at $l$. Can you see the length $IC$? (pun intended). Find $IC$.

3. The sides of a right $\triangle ABC$ have lengths $a$, $b$, and $c$. Each side is used as a chord of a circle having a radius equal to the length of the chord. Using $A_a$, $A_b$, and $A_c$ to designate the areas of the circles with radii indicated by the subscript, write an equation relating $A_a$, $A_b$, and $A_c$.

4. Two chords of a circle having lengths of 7 and 8 intersect at right angles. The chord of length 7 is partitioned into lengths of 3 and 4, while the chord of length 8 is partitioned into lengths of 2 and 6. What is the radius of the circle?
Minnesota State High School Mathematics League
Individual Event

2004-05 Event 4C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

**44.** The first two Lucas numbers are \( L_1 = 1, \ L_2 = 3 \). Successive ones are obtained from the recursion formula \( L_{n+2} = L_n + L_{n+1} \). Find the sum of the first six Lucas numbers.

**504.** \( G(1) = 3 \) and \( G(n) = \frac{4G(n)+1}{4} \) for integers \( n > 1 \). Find \( G(2005) \).

**1,625,675.**

**3.** Find the integer equal to \( \sum_{i=1}^{50} (i^3 + 1) \)

**4.** Let \( f(n) \) be the sum of the first \( n \) terms of the sequence \( 0, \ 1, \ 1, \ 2, \ 2, \ 3, \ 3, \ 4, \ 4, \ 5, \ 5, \ldots \)

For example, \( f(1) = 0, \ f(2) = 1, \ f(3) = 2, \ f(4) = 4, \) etc. Derive a formula for \( f(n) \).

Hint: your answer will take the form \( f(n) = \begin{cases} \dotfill & n \text{ even} \\ \dotfill & n \text{ odd} \end{cases} \)

4. [Barbeau, Klamkin, Moseley #46]

For \( n \) even, \( n = 2k \) and

\[
f(2k) = (0+1)+ (1+2)+ (2+3)+ \ldots + [(k-1)+k] \\
= 2[1+2+ \ldots +(k-1)] + k \\
= 2 \left( \frac{k(k-1)}{2} \right) + k \\
= k^2 - k + k \\
= k^2 \\
= \left( \frac{n}{2} \right)^2 \\
\]

For \( n \) odd, \( n = 2k-1 \) and

\[
f(2k-1) = 0 + (1+1)+ (1+2)+ \ldots + [(k-1)+(k-1)] \\
= 2 \left( \frac{k(k-1)}{2} \right) = k^2 - k \\
= \frac{(n+1)^2 - n + 1}{4} \\
= \frac{n^2 - 1}{4} \\
\]

\[
f(n) = \begin{cases} \frac{n^2}{4} & n \text{ even} \\ \frac{n^2 - 1}{4} & n \text{ odd} \end{cases} \\
\]

4. [Barbeau, Klamkin, Moseley #46]
Solutions

Minnesota State High School Mathematics League
Individual Event

2004-05 Event 4D

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

1. Find both coordinates of the focus of the parabola having directrix $y = -3$ and vertex $(3, -2)$.

2. Answer by selecting (a), (b), or (c). The line containing $A(-423, 201)$ and $B(457, -19)$ intersects the graph of $y - 204 = -\frac{83}{351}(x + 423)$
   (a) to the left of $A$
   (b) between $A$ and $B$
   (c) to the right of $B$

3. Give both coordinates of the highest point on the graph of $4x^2 + 8x + y^2 - 2y = 3$

4. The circle $x^2 + y^2 = 25$ passes through $(0, 5)$ and $(3, 4)$, and it is tangent to the line $y + 5 = 0$. Find a second circle with the same properties.

4. Let the circle be centered at $(h, k)$;
   $[\text{distance from }(0,5)]^2 = h^2 + (k-5)^2$ \hspace{1cm} (a)
   $[\text{distance from } (3,4)]^2 = (h-3)^2 + (k-4)^2$ \hspace{1cm} (b)
   $[\text{distance from } y = -5]^2 = (k+5)^2$ \hspace{1cm} (c)

   Set (a) = (c) to get $h^2 = 20k$
   Set (b) = (c) to get $h^2 - 6h = 18k$
   $20k - 6h = 18k \Rightarrow k = 3h$.
   Then $h^2 = 20(3h)$; $h = 0$ and $h = 60$
   $r = k + 5 = 185$
Minnesota State High School Mathematics League
Team Event

2004-05 Meet 4

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

1. Figure 1 shows a large circle centered at C, and a smaller circle tangent to it at A. BD = 9 and EF = 5. Find the diameter of (a) the smaller circle; (b) the larger circle.

   (a) 41
   (b) 50

\[ \sqrt{a^2 + b^2 + c^2 + d^2} \]
\[ \frac{2}{2} \]

2. Two chords of a circle intersect at right angles. The shorter chord is partitioned into lengths of a and b, while the longer chord is partitioned into lengths of c and d.
   What, in terms of a, b, c, and d, is the radius of the circle?

3. In a regular heptagonal (seven sided polygon) ABCDEFG, let \( \alpha = \angle ADB \) and set \( t = \sin \alpha \). Express \( \sin 8 \alpha \) as a polynomial of minimal possible degree in \( t \).

4. In a regular polygon ABCD... of \( n \) sides, inscribe \( \triangle ACD \). In terms of \( n \), express in degrees the measure of \( \angle ACD \).

5. Find integers \( a \) and \( b \) so that \( x^4 + 4x^3 - 2x^2 + ax + b \) is the square of a trinomial.

6. Find the largest integer \( n \) for which \( 7^n \) will divide 600! (That's 600 factorial.)

Scorers!
See the note on #2 on the solution sheet.
1. [S.I. Jones, Mathematical Nuts, #21] Let \( x = DC \); radius large 0 is 9 + x; radius small 0 is \( 9 + 2x \). Let M be the center of the small 0. (See Fig. 1 on Answer sheet.)

\[
EM^2 = CM^2 + CE^2 = \left(\frac{9 + 2x}{2}\right)^2 + \left(\frac{9 + x - 5}{2}\right)^2.
\]

Solve to get \( x = 16 \).

\( \text{diam (large 0) = 50} \); \( \text{diam (small 0) = 41} \).

2. See Fig. 2 on Ans. sheet for notation \( \angle COD = 2\beta \); \( \angle AOB = 2\alpha \).

Locate \( F \) on arc \( AB \) so \( \angle FOC = \angle AOB \).

Note that \( \angle FOD = \angle FOC + \angle COD = \angle AOB + \angle AOD \).

\( \Rightarrow 2\alpha + 2\beta = 2(\alpha + \beta) = \pi \)

**ie., FOD is a diameter; \( \Delta FCD \) is a right \( \Delta \). \( FC^2 = FC^2 + CD^2 = AB^2 + CD^2 \)

But \( AB^2 = b^2 + d^2 \) and \( CD^2 = a^2 + c^2 \).

Radius = \( \frac{1}{2} \text{FD} = \frac{1}{2} \sqrt{a^2 + b^2 + c^2 + d^2} \)

(Prob. 4 of Event 4B was a special case of this more general result.)

3. Given \( \alpha = \angle ADB \), we see from the arcs intercepted on the circumscribed circle that \( \angle BAD = 2\alpha \); \( \angle ABD = 4\alpha \).

Also, \( \alpha = \frac{\pi}{7} \).

\[
\sin 8\alpha = \sin (7\alpha + \alpha) = \sin (\pi + \alpha) = -\sin \alpha = -t
\]

4. The central angle subtending long \( \widehat{AB} \) is \( (n-3) \frac{360}{n} \), so

\[
\angle ACD = \frac{1}{2} \left( \frac{360(n-3)}{n} \right)
\]

5. The trinomial is certainly of the form \( x^2 + mx + n \).

\[
(x^2 + mx + n)^2 = x^4 + (mx)^2 + n^2 + 2mx^3 + 2nx^2 + 2mnx + (2mn)x + (n^2)
\]

\( 2m = 4, \) so \( m = 2 \)

\( m^2 + 2n = -2, \) so \( n = -3 \)

Now \( a = 2mn = -12 \)

and \( b = n^2 = 9 \)

6. The factors \( 1 \cdot 2 \cdot 4 \cdot 23 \cdot 59 \cdot 89 \).

The factors \( 1 \cdot 2 \cdot 4 \cdot 23 \cdot 59 \).

The factor \( [7^3] \) contributes 1 more 7.

\( 85 + 12 + 1 = 98 \)

**Note to Scorers re #2**

Reviewers point out that very different methods of solution give different looking but equivalent answers. The fact that \( ab = cd \) is key to establishing equivalence. Correct answers include

\[
r = \frac{\sqrt{(a + b)^2 + (c - d)^2}}{2} = \frac{\sqrt{a^2 + b^2 + c^2 + d^2}}{2}
\]

\[
= \frac{\sqrt{a^2 + c^2)(b^2 + d^2)}}{2a}
\]

\[
= \frac{\sqrt{(a^2 + c^2)(b^2 + d^2)}}{2a}
\]

\[
= \frac{(b^2 + d^2)(b^2 + c^2)}{2b}
\]