2005-06 Event 4A

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

1. Express $64^0 + 64^3 + 64^5 + 64^2$ in simplest form.

2. Express $\frac{(x^3 + y^3) - (x + y)^3}{x^2 - y^2}$ as the quotient of terms involving only the use of $x$ and $y$ to the first power.

3. Given $f(x) = \frac{\sqrt{x} + \sqrt{2}}{\sqrt{x} - \sqrt{2}}$, express $f(8) + f(32) + f(128)$ as the quotient of two relatively prime integers.

4. Find both solutions to $\sqrt{13x + 37} - \sqrt{13x - 37} = \sqrt{2}$
Minnesota State High School Mathematics League
Individual Event

2005-06 Event 4B

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

1. Two circles of radii \( r \) are placed so that the center of each lies on the circumference of the other. A line drawn through the centers \( B \) and \( C \) also intersects the circles at \( A \) and \( D \) (Figure 1). A line segment drawn from \( A \) is tangent at \( T \) to the circle centered at \( C \). Find, in terms of \( r \), the length of \( AT \).

2. In the circle in Figure 2, \( BC = 2\widehat{AB} \), \( CD = 3\widehat{AB} \), \( DA = 4\widehat{AB} \). What is the measure in degrees of \( \angle ABC \)?

3. Referring again to the two circles shown in Figure 1, what in terms of \( r \) is the area of the region that is common to the two circles?

4. In Figure 4, \( BC \) is tangent to the circle having \( AD \) as a diameter, \( AB \perp BC \), \( DC \perp BC \), \( AB = 9 \), and \( DC = 4 \). What is the area of the trapezoid \( ABCD \)?

Name ________________________ Team ________________________
2005-06 Event 4C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

1. Express $S = 128 + 64 + 32 + 16 + 8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$ as the quotient of two relatively prime integers.

2. For $n \geq 4$, find in terms of $n$ three integers $a$, $b$, and $c$ so that $a > b > c$ and $(n + 1)! - 4n! + 2(n! - 1)! = abc$.

3. Express $\sum_{k=0}^{3} (-1)^{k} \binom{4-k}{k} (xy)^{k} (x+y)^{4-2k}$ as a polynomial of five terms involving powers of $x$ and $y$.

4. The equalities

   First factor =

   Second factor =

   $1 = 1$

   $1 + \frac{1}{2} = 2(1) - \frac{1}{2}$

   $1 + \frac{1}{2} + \frac{1}{3} = 3(1) - 3\left(\frac{1}{2}\right) + \frac{1}{3}$

   $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = 4(1) - 6\left(\frac{1}{2}\right) + 4\left(\frac{1}{3}\right) - \frac{1}{4}$

   Illustrate the general equality

   $\sum_{j=1}^{n} \frac{1}{j} = \sum_{j=1}^{n} (-1)^{j+1}( )$

   Each of the missing factors on the right involves $j$ and/or $n$. What are they?

Name ___________________________ Team ___________________________
2005-06 Event 4D

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

Note: The word "curve" as used on this page may denote a straight line.

1. Find the equation of the curve that contains those points that are equidistant from the origin and the point Q(4,2).

2. Find the equation of the curve that contains those points that are equidistant from A(2,0) and the y-axis.

3. Find the equation of the curve that contains those points that are equidistant from the origin and the line $x + y = 4$.

4. Figure 4 shows a right angle, $\angle MON$, inscribed in the parabola with equation $y^2 = 4x$. The equation of the line through O and M is $y = \frac{4}{3}x$. Where does the line through M and N cut the x-axis?

Name __________________________  Team __________________________
Minnesota State High School Mathematics League
Team Event

2005-06 Meet 4

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

1. In the circle in Figure 1, \( \widehat{BC} = 2\widehat{AB} \), \( \widehat{CD} = 3\widehat{AB} \), \( \widehat{DA} = 4\widehat{AB} \). If \( AB = 1 \), then the diameter of the circle can be written in the form \( \text{diam} = \csc \alpha \). Find \( \alpha \).

2. Find both solutions to \( \sqrt{x + a} - \sqrt{x - a} = \frac{\sqrt{a}}{2} \) in terms of \( a \).

3. Express as the quotient of two relatively prime integers the product
\[
\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \ldots \left(1 - \frac{1}{2006^2}\right)
\]

4. \( (x + y)^2 - (x^2 + y^2) = k(x \chi \chi)(x^2 + xy + y^2)^2 \) Find the prime number \( k \) and the three missing factors, each of which is a linear expression in \( x \) and/or \( y \).

5. The \( p \)th term of an arithmetic progression is \( q \), and the \( q \)th term of the same progression is \( p \). What is the \((p+q)\)th term?

6. Two regular hexagons \( ABCDEF \) and \( DGHIJK \) (Figure 6) are positioned so that \( A, D, \) and \( I \) are collinear. The circle determined by \( E, D, \) and \( K \) intersects \( DI \) at \( X \). If \( AB = 4 \) and \( JK = 7 \), how long is \( AX \)?

Team __________________________

Figure 1

Figure 6