1. As the value of k changes, the equation \((2x - 3y + 6) + k(2x + y - 10) = 0\) describes a family of lines. What value of \(k\) identifies the vertical line in this family?

\(k = 3\):

Setting \(k = 3\) causes the \(y\)-terms to drop out, leaving an equation involving only \(x\).

2. Solve the system:

\[
\begin{align*}
2x - 3y + 6 &= 0 \\
2x + y - 10 &= 0
\end{align*}
\]

\[
\Rightarrow \begin{align*}
2x &= 3y - 6 \\
2x &= 10 - y
\end{align*}
\]

Equating the right sides,

\[
3y - 6 = 10 - y \Rightarrow 4y = 16 \Rightarrow y = 4.
\]

Substituting, \(2x = 3(4) - 6 \Rightarrow x = 3\).

\((x, y) = (3, 4)\).

3. The system

\[
\begin{align*}
p + q + 2r &= 1 \\
2p - q &= 3 \\
-p + 4r &= 1
\end{align*}
\]

has a unique solution. Find it.

\((p, q, r) = (1, -1, \frac{1}{2})\).

Adding the first two equations and relating the result to the third equation:

\[
\begin{align*}
3p + 2r &= 4 \\
-p + 4r &= 1
\end{align*}
\]

Now triple the bottom equation and add:

\[
14r = 7 \Rightarrow r = \frac{1}{2}.
\]

Substituting, \(-p + 4\left(\frac{1}{2}\right) = 1 \Rightarrow p = 1\), and again, \(2(1) - q = 3 \Rightarrow q = -1\).

4. Consider all possible ordered triples of integers \((x, y, z)\) which solve the system

\[
\begin{align*}
x + 3y - z &= 1 \\
3x - y - 2z &= -1
\end{align*}
\]

There are two specific single-digit positive integers, \(n\) and \(d\), such that all of the \(x\)-values in those ordered triples can be written in the form \(mn + d\), where \(m\) is any integer. Compute the values of \(n\) and \(d\).

\(n = 7\),

\(d = 4\).

Multiply the top equation by \(-2\) and add: \(x - 7y = -3 \Rightarrow y = \frac{x + 3}{7}\).

Then multiply the bottom equation by \(3\) and add: \(10x - 7z = -2 \Rightarrow z = \frac{10x + 2}{7}\).

We need \(x + 3\) and \(10x + 2\) to both be multiples of 7. This occurs when \(x = 4\), and every 7 units thereafter. So \(y\) and \(z\) will be integers when \(x\) is of the form \(7m + 4\).
1. Figure 1 shows square ABCD, of side length 2, inscribed in parallelogram AECD. If m∠F = 60°, compute the length of FC.

\[ FC = \frac{2 + \sqrt{3}}{2} \]

or \( \approx 3.155 \).

2. Figure 2 shows square ABCD, of side length 2, inscribed in kite KITE so that ΔDCE is equilateral. If angles K and T are right angles, compute the length of ET.

\[ ET = \frac{2 + \sqrt{3}}{2} \]

or \( \approx 3.732 \).

3. Figure 3 shows square ABCD, of side length 2, inscribed in trapezoid KLMN so that ΔAND is equilateral. Compute the length of NK.

\[ NK = \frac{2 + \sqrt{3}}{2} \]

or \( \approx 3.155 \).

4. A square of side length 2 is inscribed in a regular hexagon so that two sides of the square are parallel to two sides of the hexagon. Find the shortest distance between two of the parallel sides.

Drop altitudes from X and G to create similar 30-60-90 triangles, with heights x and 1. Using side ratios, \( AX = \frac{x}{\sqrt{3}} \). Using similar triangles,

\[ \frac{1}{x} = \frac{\sqrt{3}}{x} \]

But XG has the same length as the hexagon's radius \( 1 + \frac{1}{\sqrt{3}} \).

Solving \( \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} x = 1 + \frac{1}{\sqrt{3}} \), \( x = \frac{\sqrt{3} - 1}{2} \).

or \( \approx 0.366 \).
1. Compute the value of \( \sin^{-1} \left( \frac{1}{2} \right) + \cos^{-1} \left( \frac{-1}{2} \right) \).

\[
\frac{5\pi}{6}, \text{ or } 150^\circ.
\]

\[
\sin^{-1} \left( \frac{1}{2} \right) + \cos^{-1} \left( \frac{-1}{2} \right) = \frac{\pi}{6} + \frac{2\pi}{3} = \frac{5\pi}{6}.
\]

2. Express the range of the function \( f(x) = \cos (\tan^{-1} x) \). Do this by placing inequality symbols in the boxes, and real numbers, \(-\infty\), or \(+\infty\) in the blanks.

\[
0 < f(x) \leq 1.
\]

The definition of the inverse tangent function is \(-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}\), so we are looking for the values cosine can assume in quadrants I and IV and along the positive x-axis. These values are found on the interval \([0, 1]\).

3. Right triangle \(ABC\) (shown in Figure 3) has legs of lengths \(AB = 4\) and \(AC = 6\), and point \(D\) is the midpoint of side \(AC\). Compute \(\sin \angle CBD\).

\[
\Delta ABD \text{ is 3-4-5, and by Pythagoras, } BC = 2\sqrt{13}.
\]

\[
\frac{6\sqrt{13}}{65} \quad \text{or} \quad \approx 0.333.
\]

Using \(\triangle ABC\), \(\frac{\sin C}{5} = \frac{\sin \angle CBD}{3} \Rightarrow \sin \angle CBD = \frac{6}{5\sqrt{13}}.
\]

4. The side lengths of a certain triangle are three consecutive integers, and the smallest angle in the triangle has a tangent of \(\frac{2\sqrt{6}}{5}\). Find the exact value of the triangle's perimeter.

Label the three side lengths as \(a - 1\), \(a\), and \(a + 1\). The shortest side, \(a - 1\), must be opposite the smallest angle, so by the Law of Cosines:

\[
(a - 1)^2 = a^2 + (a + 1)^2 - 2a(a + 1) \cos \left( \arctan \frac{2\sqrt{6}}{5} \right) \Rightarrow \frac{5}{7} = \frac{a^2 - 2a + 1 - (a^2 + a^2 + 2a + 1)}{-2a(a + 1)}
\]

\[
\Rightarrow \frac{5}{7} = \frac{-a^2 - 4a}{-2a(a + 1)} = \frac{a + 4}{2(a + 1)} \Rightarrow 7a + 28 = 10a + 10 \Rightarrow a = 6, \text{ and } P = 3a = 18.
\]
1. If $\log_4 M = \frac{5}{2}$, then $M$ can be written in the form $2^n$. Compute $n$.

$n = \boxed{5}$.

\[
\log_4 M = \frac{5}{2} \Rightarrow M = 4^{\frac{5}{2}} = (\sqrt{4})^5 = 2^5.
\]

2. Compute the value of the sum: $\log \frac{1}{2} + \log \frac{2}{3} + \log \frac{3}{4} + \ldots + \log \frac{n}{n+1} + \ldots + \log \frac{99}{100}$.

\[
\log \frac{1}{2} + \log \frac{2}{3} + \log \frac{3}{4} + \ldots + \log \frac{n}{n+1} + \ldots + \log \frac{99}{100}
= \left( \log 1 - \log 2 \right) + \left( \log 2 - \log 3 \right) + \left( \log 3 - \log 4 \right) + \ldots + \left( \log 99 - \log 100 \right)
= \log 1 - \log 100 = 0 - 2 = -2.
\]

3. If $b = \log_3 x$, find all real values of $x$ which satisfy $\log_b \left( \log_3 x^2 \right) = 2$.

$x = \boxed{9}$. 1 point for incorrectly listing ±9; -9 is extraneous because of the definition of $b$.

\[
\log_b \left( \log_3 x^2 \right) = 2 \Rightarrow \log_b \left( 2 \cdot \log_3 x \right) = 2 \Rightarrow \log_b (2b) = 2
\Rightarrow \log_b 2 + \log_b b = \log_b 2 + 1 = 2 \Rightarrow \log_b 2 = 1, \text{ so } b = 2.
\]

Since $b = \log_3 x$, $x$ must equal 9.

4. If $\log 80 = a$ and $\log 45 = b$, write $\log 6$ as a simplified expression involving $a$ and $b$.

\[
\log 6 = \frac{a + b - 1}{2}.
\]

(This hints that $\log 60$ is exactly the average of $a$ and $b$. How does 60 relate, numerically, to 45 and 80? Hmm...)
1. Find the value of $n$ such that \[\frac{1}{1+\sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{7}} + \cdots + \frac{1}{\sqrt{2n-1} + \sqrt{2n+1}} = 100.\]

$n = 20200$. (Mathematics Teacher, August 2006)

2. The sides of a triangle have lengths $\log_8 a$, $\log_{16} a$, and $\log_{64} a$ for some value of $a$. Compute the sine of the smallest angle in this triangle.

\[\frac{\sqrt{15}}{8}, \text{ or } \approx 0.484\]

3. Figure 3 shows square $ABCD$, of side length 2, inscribed in trapezoid $KLMN$ so that $\triangle AND$ is equilateral. If $m\angle M = \alpha$, express the length of $LM$ in terms of $\sin \alpha$.

$LM = \frac{1+\sqrt{3}}{\sin \alpha}$

4. Two people, both starting at sunrise, walk toward each other from opposite ends of a hiking trail. They meet at noon. Both continue hiking, the faster one finishing the trail at 5:20 pm, the slower one finishing at 8:20 pm. What time was sunrise?

5:20 am.

5. Figure 5 shows square $ABCD$, of side length 2, inscribed in cyclic quadrilateral $CRLE$ so that $\triangle ALD$ is equilateral. Compute the length of diagonal $LC$.

$LC = \sqrt{2 + \sqrt{6}}, \text{ or } \approx 3.864$

6. Alice, Beth, and Cindy, starting together, walk in the same direction around a circular track. It takes the girls $\frac{5}{36}$, $\frac{2}{9}$, and $\frac{35}{99}$ of an hour, respectively, to walk once around the track. How many laps will Alice have walked by the next time all three girls are together again at the starting point?

56 laps.
1. Multiply the top and bottom of each fractional term by the conjugate of its denominator:

\[
\frac{1}{1+\sqrt{3}} \cdot \frac{1-\sqrt{3}}{1-\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} \cdot \frac{\sqrt{3}-\sqrt{5}}{\sqrt{3}-\sqrt{5}} + \frac{1}{\sqrt{2n-1}+\sqrt{2n+1}} \cdot \frac{\sqrt{2n-1}-\sqrt{2n+1}}{\sqrt{2n-1}-\sqrt{2n+1}} = 100
\]

\[
\frac{1-\sqrt{3}}{-3} + \frac{\sqrt{5}-\sqrt{5}}{-5} + \frac{\sqrt{7-\sqrt{7}}}{5-7} + \frac{\sqrt{2n-1}-\sqrt{2n+1}}{-2n-1-2n+1} = 100
\]

So \(\sqrt{2n+1} = 201 \Rightarrow 2n+1 = 201^2 \Rightarrow 2n = (201-1)(201+1) \Rightarrow n = (200)(101) = 20200.

2. Using change of base, \(\log_8 a = \frac{\log_4 a}{\log_2 8} = \frac{1}{3} \log_2 a\). Similarly, \(\log_{16} a = \frac{1}{4} \log_2 a\), and \(\log_{64} a = \frac{1}{6} \log_2 a\).

So this triangle is similar to one with side lengths \(\frac{1}{3}, \frac{1}{4}, \text{and } \frac{1}{6}\). Multiplying by a scale factor of the LCD (12) produces a triangle with integer side lengths: 4, 3, 2. Call the smallest angle \(\angle S\). By the Law of Cosines, \(2^2 = 3^2 + 4^2 - 2(3)(4)\cos S \Rightarrow \cos S = \frac{9-16}{-24} = \frac{7}{8}\), and \(\sin S = \sqrt{1-\cos^2 S} = \frac{\sqrt{15}}{8}\).

3. Chasing angles, \(\angle CDM = 30^\circ\) and \(\angle CBL = 60^\circ\). Label \(CM = x\) and \(CL = y\). By the Law of Sines,

\[
\frac{\sin 30^\circ}{x} = \frac{\sin \alpha}{2} \Rightarrow x \sin \alpha = 1 \Rightarrow x = \frac{1}{\sin \alpha}, \text{ and } \frac{\sin 60^\circ}{y} = \frac{\sin (180^\circ - \alpha)}{2} \Rightarrow y \sin \alpha = \sqrt{3}
\]

\[\Rightarrow y = \frac{\sqrt{3}}{\sin \alpha}\]. So \(LM = CM + CL = x + y = \frac{1+\sqrt{3}}{\sin \alpha}\).

4. Let \(s\) be the speed of the slow walker, \(f\) the speed of the fast walker, and \(t\) the number of hours between sunrise and noon. The distance that the fast walker travels before noon is the same distance that the slow walker travels after noon, so \(ft = \frac{25}{3}s\). Also, the distance that the slow walker travels before noon is the same distance that the fast walker travels after noon, so \(st = \frac{16}{3}f\). Multiplying these two equations yields

\[fst^2 = \frac{25}{3} \cdot \frac{16}{3} \cdot f \Rightarrow t^2 = \frac{25 \cdot 16}{9} \Rightarrow t = \frac{5 \cdot 4}{3} = \frac{62}{3} \text{ hours. Sunrise occurred at } 5:20 \text{ am.}\]

5. Chasing angles, \(\angle EAB = 30^\circ\) and \(\angle ECD = 30^\circ\). Various 30-60-90 relationships allow us to label the perimeter of CRLE as shown. Note that \(\triangle LER \cong \triangle LEC\), so diagonals \(CL\) and \(ER\) are congruent! Because CRLE is cyclic, we apply Ptolemy's Theorem: \(EL \cdot CR + LR \cdot EC = LC \cdot ER\)

\[\Rightarrow \left(2 + \frac{4}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right) + \left(2 + \frac{2}{\sqrt{3}}\right)^2 = \frac{4 + \sqrt{3}}{3} + \left(4 + \frac{8}{\sqrt{3}} + \frac{4}{3}\right) = \frac{12}{\sqrt{3}} + 8 = 4\sqrt{3} + 8 = CL^2\]. Hoping \(4\sqrt{3} + 8\) is a perfect square, set \(CL = a + b\sqrt{3}\) and solve the resulting system: \(a = \sqrt{2}\), \(b = \sqrt{2} \Rightarrow CL = \sqrt{2 + \sqrt{6}}\).

6. First find the LCD of the three rates: \(36 = 2^2 \cdot 3^2\), \(9 = 3^2\), \(99 = 3^2 \cdot 11\) \(\Rightarrow LCD = 2^2 \cdot 3^2 \cdot 11 = 396\).

\[r_A = \frac{55}{396}, \ r_B = \frac{88}{396}, \ r_C = \frac{140}{396}\]. Now examine the numerators: \(55 = 5 \cdot 11\), \(88 = 2^3 \cdot 11\), \(140 = 2^2 \cdot 5 \cdot 7\). What factors is Alice's numerator missing? \(2^3 \cdot 7 = \boxed{56} \text{ laps}\). (Why does this work?)