Minnesota State High School Mathematics League
2010-11 Meet 2, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

\[ x = \frac{14}{15} \]

1. Given \( \frac{4}{9} x = \frac{14}{15} \), determine \( x \) exactly.

\[ m = \frac{25}{18} \]

2. In solving the equation \( \frac{2}{3} \left( x - \frac{5}{4} \right) = \frac{3}{4} x + \frac{5}{9} \), Pat eventually arrived at the step which looked like \( \frac{25}{18} = mx \). Determine exactly the value of \( m \).

\[ c = \frac{a}{b} - cx = \frac{a}{b} - bx \]

3. At a school dance attended by 200 students, Anna danced with 7 boys, Beth danced with 8 boys, Carmen danced with 9 boys, and so on through the last girl, who danced with all of the boys. How many boys attended the dance?

4. Mrs. Witquick is creating a problem to assign to her algebra class. The problem is of the form \( \frac{a}{b} - cx = \frac{a}{b} - bx \), where \( a, b, \) and \( c \) are distinct non-zero real numbers.

If Mrs. W wants the solution (for \( x \)) to be the reciprocal of \( a \), what value must \( c \) have (in terms of \( a \) and \( b \)) to generate the desired solution?
1. Given \( \frac{4}{9} x = \frac{14}{15} \), determine \( x \) exactly.

\[
\frac{4}{9} x = \frac{14}{15} \quad \Rightarrow \quad x = \frac{14}{15} \cdot \frac{9}{4} \quad \Rightarrow \quad x = \frac{3}{2} \cdot \frac{7}{5} = \frac{21}{10}.
\]

2. In solving the equation \( \frac{2}{3} \left( x - \frac{5}{4} \right) = \frac{3}{4} x + \frac{5}{9} \), Pat eventually arrived at the step which looked like \( -\frac{25}{18} = mx \). Determine exactly the value of \( m \).

\[
\frac{2}{3} \left( x - \frac{5}{4} \right) = \frac{3}{4} x + \frac{5}{9} \quad \Rightarrow \quad \frac{2}{3} x - \frac{10}{6} = \frac{3}{4} x + \frac{5}{9} \quad \Rightarrow \quad -\frac{15}{18} = -\frac{10}{12} = \frac{9}{12} x - \frac{8}{12} x \quad \Rightarrow \quad -\frac{25}{18} = \frac{1}{12} x.
\]

3. At a school dance attended by 200 students, Anna danced with 7 boys, Beth danced with 8 boys, Carmen danced with 9 boys, and so on through the last girl, who danced with all of the boys. How many boys attended the dance?

The first girl danced with \( 6 + 1 \) boys, the second girl danced with \( 6 + 2 \) boys, etc., so we can say that the \( n^{th} \) girl danced with \( 6 + n \) boys. Since there were \( n \) girls and \( 6 + n \) boys at the dance,

\[
n + (6 + n) = 200 \quad \Rightarrow \quad 2n + 6 = 200 \quad \Rightarrow \quad n = 97, \quad \text{so there were 97 girls and 103 boys.}
\]

4. Mrs. Witquick is creating a problem to assign to her algebra class. The problem is of the form \( \frac{a}{b} - cx = \frac{a}{c} - bx \), where \( a, b, \) and \( c \) are distinct non-zero real numbers.

If Mrs. W wants the solution (for \( x \)) to be the reciprocal of \( a \), what value must \( c \) have (in terms of \( a \) and \( b \)) to generate the desired solution?

Avoid the temptation to substitute \( \frac{1}{a} \) for \( x \) immediately. Instead, solve the equation for \( x \):

\[
\frac{a}{b} - cx = \frac{a}{c} - bx \quad \Rightarrow \quad bx - cx = \frac{a}{c} - \frac{a}{b} \quad \Rightarrow \quad x(b - c) = \frac{ab - ac}{bc} \quad \Rightarrow \quad x(b - c) = \frac{ab - ac}{bc}
\]

Since \( b \neq c \), divide both sides by \( b - c \), leaving \( x = \frac{a}{bc} \). Since \( x = \frac{1}{a} \), \( \frac{1}{a} = \frac{a}{bc} \Rightarrow c = \frac{a^2}{b} \).
1. In $\triangle ABC$ (Figure 1), if all of the medians, altitudes, and angle bisectors are drawn, the **shortest** of these segments will be the (altitude) (angle bisector) (median) [circle one] to segment (AB) (AC) (BC) [circle one].

\[ AD = \_\_\_\_\_\_\_\_ \]

2. Again using $\triangle ABC$ (Figure 1), draw the angle bisector from point $B$ to side $\overline{AC}$. Where the angle bisector intersects $\overline{AC}$, call that point $D$.

Determine exactly the length of $\overline{AD}$. 

\[ \_\_\_\_\_\_\_\_ \]

3. Again using $\triangle ABC$ (Figure 1), determine exactly the length of the median drawn from point $B$ to side $\overline{AC}$.

\[ \_\_\_\_\_\_\_\_ \]

4. Calculate the surface area of the triangular pyramid formed using $\triangle ABC$ (Figure 1) as the base and a point 12 units directly "above" point $B$ as the apex.

(In this case, "above" means that the segment connecting the apex and $B$ would be perpendicular to the plane of this paper)

\[ \_\_\_\_\_\_\_\_ \]

Name: _____________________________  Team: _____________________________
1. In \( \triangle ABC \) (Figure 1), if all of the medians, altitudes, and angle bisectors are drawn, the **shortest** of these segments will be the:

- (altitude)
- (angle bisector)
- (median) [circle one]

   **[To segment (AB) (AC) (BC) [circle one].]**

   Since the shortest distance from a point to a line is along a perpendicular to the line, the shortest segment must be an **altitude**. Furthermore, the shortest altitude will emanate from the largest angle: located at B and drawn to \( \overline{AC} \).

   \[
   AD = \frac{30}{7} \]
   
   or \( \approx 4.286 \)

2. Again using \( \triangle ABC \) (Figure 1), draw the angle bisector from point B to side \( \overline{AC} \). Where the angle bisector intersects \( \overline{AC} \), call that point D.

   Determine exactly the length of \( AD \).

   **Label length** \( AD \) as \( x \). Then \( CD = 12 - x \). By the Angle Bisector Theorem,

   \[
   \frac{5}{9} = \frac{x}{12-x} \quad \Rightarrow \quad 5(12-x) = 9x \quad \Rightarrow \quad 60 = 14x \quad \Rightarrow \quad x = \frac{30}{7}.
   \]

   \[\sqrt{17} \]

   or \( \approx 4.123 \)

3. Again using \( \triangle ABC \) (Figure 1), determine exactly the length of the median drawn from point B to side \( \overline{AC} \).

   **Label the median as length** \( x \) (see Figure 3). Applying Stewart’s Theorem,

   \[
   5^2 \cdot \beta + 9^2 \cdot \beta = x^2 \cdot 2 + \beta \cdot 6 \cdot 12 \quad \Rightarrow \quad 2x^2 = 25 + 81 - 72 \quad \Rightarrow \quad x = \sqrt{17}.
   \]

4. Calculate the surface area of the triangular pyramid formed using \( \triangle ABC \) (Figure 1) as the base and a point 12 units directly "above" point B as the apex. (In this case, "above" means that the segment connecting the apex and B would be perpendicular to the plane of this paper)

   **Call the new point** \( P \) (see Figure 4). Right triangles \( PAB \) and \( PCB \) conveniently conform to the Pythagorean triples 5-12-13 and 9-12-15 respectively. We can find their areas using the basic triangle area formula:

   \[
   \text{Area}[\triangle PAB] = \frac{1}{2} \cdot 5 \cdot 12 = 30 \quad \text{Area}[\triangle PCB] = \frac{1}{2} \cdot 9 \cdot 12 = 54
   \]

   We then use Heron’s Formula on triangles \( PAC \) and \( ABC \):

   \[
   \text{Area}[\triangle PAC] = \sqrt{20(8)\cdot(7)\cdot(5)} = \sqrt{400 \cdot 14} = 20\sqrt{14}
   \]

   \[
   \text{Area}[\triangle ABC] = \sqrt{13(8)\cdot(4)\cdot(1)} = \sqrt{16 \cdot 26} = 4\sqrt{26}
   \]

   So the total surface area is \( 84 + 20\sqrt{14} + 4\sqrt{26} \).
Minnesota State High School Mathematics League
2010-11 Meet 2, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS are allowed on this event.

θ = ___________ 1. Determine the acute angle θ such that \( \sin \theta = 2 \cos 57^\circ \sin 57^\circ \).

\[ \cos \angle A = \quad \] 2. Triangle \( ABC \) is isosceles, with base \( BC = 10 \) and legs \( AB = AC = 13 \).
Determine \( \cos \angle A \) exactly.

\[ \sin \angle R = \quad \] 3. In quadrilateral \( PQRS \), \( PQ = 3 \), \( QR = 5 \), and \( RS = PS = PR = 4 \).
Determine \( \sin \angle R \) exactly.

\[ \sin 2\beta = \quad \] 4. Acute angles \( \alpha \) and \( \beta \) satisfy \( 3 \sin(\alpha + \beta) - 4 \sin \alpha = 4 \cos \alpha - 3 \cos(\alpha - \beta) \).
Determine \( \sin 2\beta \) exactly.

Name: ___________________________  Team: ___________________________
SOLUTIONS

1. Determine the acute angle $\theta$ such that $\sin \theta = 2 \cos 57^\circ \sin 57^\circ$.

   By double-angle identity, $2 \cos 57^\circ \sin 57^\circ = \sin 2(57^\circ) = \sin 114^\circ$, and a quick reflection across the y-axis (Figure 1) shows that the acute angle $(180^\circ - 114^\circ) = 66^\circ$ has the same sine.

2. Triangle $ABC$ is isosceles, with base $BC = 10$ and legs $AB = AC = 13$.

   Determine $\cos \angle A$ exactly.

   Dropping the altitude from $A$ divides $\triangle ABC$ into two 5-12-13 triangles: 
   \[
   \cos A = \cos 2(\angle BAD) = 1 - 2 \cdot \sin^2 (\angle BAD)
   \]
   \[
   = 1 - 2 \cdot \left(\frac{5}{13}\right)^2 = 1 - 2 \cdot \frac{25}{169} = \frac{119}{169}.
   \]


   Determine $\sin \angle R$ exactly.

   Drawing a good diagram is half the battle here (see Figure 3).
   \[
   \sin \angle R = \sin (\angle QRP + 60^\circ) = \sin \angle QRP \cdot \cos 60^\circ + \cos \angle QRP \cdot \sin 60^\circ
   \]
   \[
   = \left(\frac{3}{5} \cdot \frac{1}{2}\right) + \left(\frac{4}{5} \cdot \frac{\sqrt{3}}{2}\right) = \frac{3 + 4\sqrt{3}}{10}.
   \]

4. Acute angles $\alpha$ and $\beta$ satisfy $3 \sin(\alpha + \beta) - 4 \sin \alpha = 4 \cos \alpha - 3 \cos(\alpha - \beta)$.

   Determine $\sin 2\beta$ exactly.

   Rearrange the original equation to read: 
   \[3 \sin (\alpha + \beta) + 3 \cos (\alpha - \beta) = 4 \sin \alpha + 4 \cos \alpha\]
   So 
   \[3 \left[ \sin \alpha \cos \beta + \cos \alpha \sin \beta + \cos \alpha \cos \beta + \sin \alpha \sin \beta \right] = 4 \left( \sin \alpha + \cos \alpha \right)\]
   Factoring the LHS by grouping, 
   \[3 \left( \sin \alpha + \cos \alpha \right) \left( \sin \beta + \cos \beta \right) = 4 \left( \sin \alpha + \cos \alpha \right) \left( \sin \beta + \cos \beta \right) \Rightarrow \left( \sin \alpha + \cos \alpha \right) \left[ 3 \left( \sin \beta + \cos \beta \right) - 4 \right] = 0\]
   So either 
   \[
   \sin \alpha + \cos \alpha = 0, \text{ which is impossible for two acute angles, or } \sin \beta + \cos \beta = \frac{4}{3}.
   \]
   Then 
   \[
   \left( \sin \beta + \cos \beta \right)^2 = \sin^2 \beta + 2 \sin \beta \cos \beta + \cos^2 \beta = 1 + 2 \sin 2\beta = \frac{16}{9}, \text{ so } \sin 2\beta = \frac{7}{9}.
   \]
Minnesota State High School Mathematics League
2010-11 Meet 2, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 12 minutes for this event.

1. On the fictional Xtreme and Yowzer temperature scales, the values are related by a linear equation. If 300° Xtreme corresponds to 70° Yowzer, and 0° Xtreme corresponds to 20° Yowzer, at what temperature will both scales read the same?

\[ a = \quad \] 2. If the graphs of \[ 2y + x + 3 = 0 \] and \[ 3y + ax + 2 = 0 \] meet at right angles, then what is the value of \( a \)?

3. Consider the line \( \ell \) containing the points (-3, 7) and (6, -2). Determine exactly the length of the hypotenuse of the right triangle formed by the intersection of \( \ell \) and the \( x \)- and \( y \)-axes.

4. Calculate the sum of the radii of all circles that are tangent to the \( x \)-axis and pass through the points \( A \) (1, 9) and \( B \) (8, 8).

Name: __________________________ Team: __________________________
1. On the fictional Xtreme and Yowzer temperature scales, the values are related by a linear equation. If 300° Xtreme corresponds to 70° Yowzer, and 0° Xtreme corresponds to 20° Yowzer, at what temperature will both scales read the same?

Suppose the linear equation is \( Y = mX + b \). Then the "Yowzer"-intercept is 20, and

\[ 70 = m(300) + 20 \implies m = \frac{1}{6}. \]

Letting \( Y = X \),

\[ X = \frac{5}{6} \times 20 \implies X = 24°. \]

2. If the graphs of \( 2y + x + 3 = 0 \) and \( 3y + ax + 2 = 0 \) meet at right angles, then what is the value of \( a \)?

[Mathematics Teacher, 2006]

\[ 2y + x + 3 = 0 \implies y = -\frac{1}{2}x - \frac{3}{2}, \text{ which means that the perpendicular } 3y + ax + 2 = 0 \text{ has a slope of 2.} \]

\[ 3y + ax + 2 = 0 \implies y = -\frac{a}{3}x - \frac{2}{3}, \text{ so } -\frac{a}{3} = 2, \text{ and } a = -6. \]

3. Consider the line \( \ell \) containing the points \((-3, 7)\) and \((6, -2)\). Determine exactly the length of the hypotenuse of the right triangle formed by the intersection of \( \ell \) and the \(x\)- and \(y\)-axes.

\( \ell \) has slope \( \frac{-2-7}{6-(-3)} = \frac{-9}{9} = -1 \), from which we generate the equation \( y = -x + b \). Substituting the point \((-3, 7)\) yields \( 7 = -(3) + b \implies b = 4 \). The intercepts of \( y = -x + 4 \) are \((0, 4)\) and \((4, 0)\), forming a 45-45-90 triangle with the axes. Its legs are length 4, hypotenuse \( 4\sqrt{2} \).

4. Calculate the sum of the radii of all circles that are tangent to the \(x\)-axis and pass through the points \(A\ (1, 9)\) and \(B\ (8, 8)\).

[Mathematics Teacher, 2006]

The centers of these circles must lie on the perpendicular bisector of \( \overline{AB} \). The midpoint of \( \overline{AB} \) is \((4.5, 8.5)\), and the slope of \( \overline{AB} \) is \(-\frac{1}{7}\), so the perpendicular bisector has slope 7 and equation \( y - 8.5 = 7(x - 4.5) \implies y = 7x - 23 \). The distance from any point on this line to the \(x\)-axis is \( |7x - 23| \), and the distance to point \(A\) is \( \sqrt{(x - 1)^2 + (7x - 23 - 9)^2} \). Since these distances are both radii, set them equal and square: \( (x - 1)^2 + (7x - 32)^2 = (7x - 23)^2 \implies x^2 - 128x + 496 = 0 \). Solutions are \(x = 4 \) or 124, yielding radii of lengths 5 and 845. The sum of those lengths is 850.
Minnesota State High School Mathematics League
2010-11 Meet 2, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

___________ 1. In \( \triangle ABC \) (Figure 1), determine exactly the length of the altitude drawn from \( B \) to side \( \overline{AC} \).

\[ m = \]

___________ 2. Calculate the greatest value of \( m \) such that the graph of \( y = mx \) intersects the graph of \( (x-10)^2 + (y-5)^2 = 4 \).

___________ 3. In the days when hay was cut by hand, a team of workers set out to cut two fields, one twice the area of the other. The first half day, the entire team worked in the larger field. Then the team split in half, one half finishing the large field by the end of the day, the other half working in the smaller field. At the end of the day, there was still a little work left in the smaller field, but one man finished it up working all of the next day. How many men were on the team?

\[ \sin \angle ABC = \]

___________ 4. In quadrilateral \( PQRS \), \( PQ = QR = SQ = 10 \) and \( RS = PS = 16 \). Determine exactly the sine of the interior angle located at \( Q \).

___________ 5. A trained cat and dog run a race on a straight track 100 feet long. From the starting line at one end of the track, each must cross a line at the other end of the track, and then return across the line from which they started. The dog leaps three feet at each bound. The cat covers only two feet at a leap, but makes three leaps to the dog’s two. Who wins, and how many feet short of the finish line is the loser? (Must answer both questions correctly)

\[ \theta = \]

___________ 6. Determine, in degrees, the smallest positive angle \( \theta \) for which \( \cos \frac{\theta}{2} = \frac{1 - \sin 4\theta}{2} \).

Team: _______________________________
1. In \( \triangle ABC \) (Figure 1), determine exactly the length of the altitude drawn from \( B \) to side \( AC \).

\[
\frac{2 \sqrt{26}}{3}
\]

2. Calculate the greatest value of \( m \) such that the graph of \( y = mx \) intersects the graph of \( (x - 10)^2 + (y - 5)^2 = 4 \).

[\textit{Mathematics Teacher, 2006}]

3. In the days when hay was cut by hand, a team of workers set out to cut two fields, one twice the area of the other. The first half day, the entire team worked in the larger field. Then the team split in half, one half finishing the large field by the end of the day, the other half working in the smaller field. At the end of the day, there was still a little work left in the smaller field, but one man finished it up working all of the next day. How many men were on the team?

\[
\sin \angle PQR = \frac{-336}{625}
\]

or \(-0.5376\)

4. In quadrilateral \( PQRS \), \( PQ = QR = SQ = 10 \) and \( RS = PS = 16 \). Determine exactly the sine of the interior angle located at \( Q \).

5. A trained cat and dog run a race on a straight track 100 feet long. From the starting line at one end of the track, each must cross a line at the other end of the track, and then return across the line from which they started. The dog leaps three feet at each bound. The cat covers only two feet at a leap, but makes three leaps to the dog’s two. Who wins, and how many feet short of the finish line is the loser?  \( \text{\textit{Must answer both questions correctly}} \)

\[
\theta = 54^\circ
\]

6. Determine, in degrees, the smallest positive angle \( \theta \) for which \( \cos \frac{\theta}{2} = \sqrt{\frac{1 - \sin 4\theta}{2}} \).
1. Label Figure 1 as shown. Then \((25 - x^2) + (12 - x)^2 = 9^2 \Rightarrow 25 - x^2 + 144 - 24x + x^2 = 81 \Rightarrow 88 = 24x\)

\[ x = \frac{11}{3} \]  

The altitude has length \(\sqrt{25 - x^2} = \sqrt{25 - \left(\frac{11}{3}\right)^2} = \sqrt{\frac{104}{9}} = \frac{2\sqrt{26}}{3}\).

2. Substitute \(mx\) for \(y\) to obtain the equation \((x-10)^2 + (mx-5)^2 = 4\). Expanding, combining like terms, and clearing the right side of the equation results in the quadratic \((m^2+1)x^2 - (10m+20)x + 121 = 0\). Since we are looking for an extreme value of \(m\) (the "greatest" value that will achieve an intersection), and we wish for a line to intersect a circle, this extreme value will cause the line to be tangent to the circle; i.e. the above quadratic will have only one solution. Set the discriminant equal to zero: \((10m+20)^2 - 4(m^2+1)(121) = 0 \Rightarrow -384m^2 + 400m - 84 = 0\), the greatest solution of which is \(\frac{3}{4}\).

3. Since the large field was cut in a full team’s half-day plus a half-team’s half-day, the full team cut \(\frac{2}{3}\) of the large field in its half-day. Then the half-team that moved to the smaller field must have cut \(\frac{2}{3}\) of that field, leaving an area equal to \(\frac{1}{6}\) of the large field to be cut the next day. That took one man all day. Back to the first day: with everyone working, a total area equal to \(\frac{4}{3}\) of the large field was cut, so there must have been \(\frac{4}{\frac{1}{6}} = 8\) men on the team.

(This can, of course, also be worked in the more traditional way of letting \(x\) be the number of workers, etc.)

[This is an Americanization of a favorite problem of Russian author Leo Tolstoy]

4. As shown in Figure 4, the quadrilateral is concave, and can be sectioned into four 6-8-10 right triangles. Therefore, the interior angle at \(Q\) is comprised of four identical angles: \(\sin \angle PQR = \sin \left(4 \cdot \angle PQT\right) = 2 \cdot \sin \left(2 \cdot \angle PQT\right) \cdot \cos \left(2 \cdot \angle PQT\right)\)

\[
= 4 \cdot \sin \angle PQT \cdot \cos \angle PQT \cdot \left(2 \cos^2 \angle PQT - 1\right) = 4 \left(\frac{4}{5}\right) \left(\frac{3}{5}\right) \left[2 \cdot \left(\frac{3}{5}\right)^2 - 1\right] = \frac{48}{25} \cdot \frac{-7}{25} = -\frac{336}{625}.
\]

5. The cat needs 100 leaps to go 200 feet; the dog, having made 33 leaps to go 99 feet, needs a 34th to pass the end of the track, and therefore 34 more leaps to get back, for a total of 68. But in the time it takes the cat to make 100 leaps, the dog only makes 66, ending up 2 leaps short of the finish line. The cat wins, by 2 feet. [Mathematical Puzzles of Sam Loyd]

6. Rewrite the LHS using the half-angle formula for cosine: \(\cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}} = \sqrt{\frac{1 - \sin 4\theta}{2}}\). Squaring both sides and simplifying yields \(\cos \theta = -\sin 4\theta\). The first range of values where this could be true is \(45^\circ \leq \theta \leq 90^\circ\), so that \(180^\circ \leq 4\theta \leq 360^\circ\) (making \(\sin 4\theta\) negative). Now note that \(-\sin 4\theta = \sin (-4\theta)\), so that \(\cos \theta = \sin (-4\theta)\).

\(4\theta\) is smallest in Quadrant III, forcing \(-4\theta\) into Quadrant II, with \(\theta + 90^\circ = (-4\theta) + 360^\circ \Rightarrow 5\theta = 270^\circ \Rightarrow \theta = 54^\circ\).