Solutions

Minnesota State High School Mathematics League
Individual Event

2003-04 Event 2A

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

1. Solve for \(x\): \[ \frac{2}{3} + \frac{1}{6} = \frac{1}{x} \]

\[ \frac{4 + 1}{6} = \frac{1}{x} \]

\[ \frac{5}{6} = \frac{1}{x} \]

\[ x = \frac{6}{5} \]

2. Solve for \(x\) and simplify:
\[ ax + \frac{a}{a+b} = bx + \frac{b}{a+b} \]

\[ \frac{-1}{a+b} \]

3. Find all \(x\) that satisfy \(|x - 3| = \frac{1}{2}x\)

\[ 2, 6 \]

4. My bathroom scale is not zeroed correctly, but otherwise works fine. When I stand on it, it registers 169 pounds; when my grandson stands on it, it registers 38. When I stand on it holding my grandson, it registers 209. How much do I actually weigh?

\[ 4. \text{ Let the error be } e. \]

My weight = 169 + e
Grandson's wt = 38 + e
Combined wt = 209 + e

\[(169 + e) + (38 + e) = 209 + e \]

\[ 207 + e = 209 \]

\[ e = 2 \]

My weight = 171
2003-04 Event 2B

1. The base of a parallelogram $ABCD$ is $AB = 7$. From vertex $D$, an altitude of length $4$ is dropped to a point $E$ that is three units from $A$ (Figure 1). What is the perimeter of parallelogram $ABCD$?

2. Figure 2 shows a $5$-$12$-$13$ right $\triangle ABC$ with a perpendicular dropped from the right angle at $C$ to $D$ on $AB$. Express the perimeter of $\triangle ADC$ as the quotient of two relatively prime integers.

3. Lines drawn from the center $O$ of a regular hexagon to its six vertices partition the hexagon into six equilateral triangles. The incenters of these triangles (that is, the points where the angle bisectors are concurrent) are then joined to form a smaller regular hexagon. If the larger hexagon has a perimeter of $6$, express in exact form the perimeter of the smaller one?

4. A square of side length $1$ is inscribed as shown in Figure 4 in a right $\triangle ABC$ that has $\angle ABC = 30^\circ$. What is the perimeter of $\triangle ABC$, expressed in the form $\frac{a\sqrt{b} + c}{d}$ where $a, b, c,$ and $d$ are integers.

\[ \frac{2 \sqrt{3}}{2} \]
With \(0 < a < b\), Figure 1 shows two angles \(\alpha\) and \(\beta\) drawn in the unit circle so that \(\sin \alpha = a, \sin \beta = -b\).

Express the following in terms of \(a\) and \(b\). Pay attention to signs.

1. \(\cos \alpha = \frac{-\sqrt{1-a^2}}{a}\)

2. \(\sin(\alpha + \beta) = -a \sqrt{1-b^2} + b \sqrt{1-a^2}\)

3. \(\tan \left(\frac{\pi}{2} + \alpha\right) = \frac{\sqrt{1-a^2}}{a}\)

4. \(\cos 4\alpha = 1 - 8a^2 + 8a^4\)

4. Since \(\cos 2\theta = 2\cos^2 \theta - 1\),
   \(\cos 4\alpha = \cos 2(2\alpha)\)
   \(= 2\cos^2 2\alpha - 1\)
   \(= 2\left[2\cos^2 \alpha - 1\right] - 1\)
   \(= 2\left[4\cos^4 \alpha - 4\cos^2 \alpha + 1\right] - 1\)
   \(= 8(\cos^2 \alpha)^2 - 8\cos^2 \alpha + 1\)
   \(= 8(1-a^2)^2 - 8(1-a^2) + 1\)
   \(= 1 - 8a^2 + 8a^4\)

3. As is clear from the picture at the right, for an angle \(\theta\), \(0 < \theta < \frac{\pi}{2}\), \(\tan \left(\frac{\pi}{2} + \theta\right) = \frac{s}{r} = -\cot \theta\).
   But such formulas remain true for any \(\theta\).
   \(\therefore \tan \left(\frac{\pi}{2} + \alpha\right) = -\cot \alpha = -\frac{\cos \alpha}{\sin \alpha} = -\frac{\sqrt{1-a^2}}{a}\)
Solutions

Minnesota State High School Mathematics League
Individual Event

2003-04 Event 2D

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

1. The graph of the line \(3x + 2y = 6\), together with the positive coordinate axes, forms a triangle. Find the area of the triangle.

2. Figure 2 shows a line tangent at (-2,3) to a circle centered at the origin. Where does this line cross the y-axis?

3. A circle centered at (2,7) with a radius of 5 and a circle centered at (0,1) with a radius of \(\sqrt{5}\) intersect in points \(A\) and \(B\). Where does the line through \(A\) and \(B\) cut the y-axis?

4. A line \(L\) through (-2,3) and a line through the origin intersect so that with the positive x-axis as base, an isosceles triangle is formed. If the area of the triangle is \(\frac{25}{4}\), what is the equation of \(L\)?

\[
y - 3 = -\frac{1}{2} (x + 2) \quad \text{or} \quad x + 4y - 10 = 0
\]

3. The two circles are
\[
x^2 - 4x + y^2 - 14y + 28 = 0
\]
\[
x^2 + y^2 - 2y - 4 = 0
\]
Subtract;
\[
-4x - 12y + 32 = 0
\]
This line contains \(A\) and \(B\).
When \(x = 0\), \(y = \frac{8}{3}\)

$\text{Area} = \frac{1}{2} (2)(3) = 3$

Let the line through the origin have equation \(y = mx\). Then \(L\) has the equation
\[
y - 3 = -m (x + 2) \quad \text{Thus,}
\]
\[
y - 3 = -y - 2m \quad y = \frac{3 - 2m}{2}
\]
The lines intersect where
\[
y = \frac{3 - 2m}{2} \quad \text{and} \quad x = \frac{3 - 2m}{2m}
\]
The area is
\[
\text{Area} = \frac{1}{2} 2 \times \frac{25}{4m} = \frac{25}{4m (m - 9)} = 0; \quad m \neq 9; \quad m = \frac{1}{4}
\]
Answers

Minnesota State High School Mathematics League
Team Event

2003-04 Meet 2

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

1. In Event 2D, we found a line through \((-2,3)\) and a line \(y = mx\) that intersected at a point \(P\) to form with the positive x-axis an isosceles triangle. Find an equation in the form \(y = f(x)\) of the curve containing all such points \(P\).

\[
y = \frac{3x}{2x+2}
\]

2. The hypotenuses of two right triangles, \(\triangle ABC\) and \(\triangle ABD\) (Figure 2), intersect a \(E\), and from \(E\) a perpendicular is dropped to \(F\) on \(AB\). Let \(a = AC\), \(b = BD\), and \(c = EF\). Express \(c\) in terms of \(a\) and \(b\).

\[
c = \frac{ab}{a+b}
\]

3. Find all \(x\) that satisfy \(|x + 2| = \frac{1}{2}x + 3\).

\[
x - \frac{10}{3}
\]

4. Write in standard form (i.e. \(ax + by + c = 0\)) the equation of one of the two lines that are tangent to the circle \(x^2 + y^2 = 36\) and make with the positive coordinate axes a triangle of area 60.

\[
3x + y - 6\sqrt{10} = 0 \quad \text{or} \quad x + 3y - 6\sqrt{10} = 0
\]

5. Figure 5 shows the graph of a parabola \(y = x^2 + bx + c\) having its lowest point at \((m,n)\), \(n > 0\). Express the roots of \(x^2 + bx + c = 0\) in terms of \(m\) and \(n\). In expressing your answer, don't leave negative numbers under a radical sign.

\[
m \pm i\sqrt{n}
\]

6. A point \(P\) is located interior to a rectangle \(ABCD\) so that its distances from \(A\), \(B\), and \(C\) are given by \(AP = \sqrt{74}\), \(BP = \sqrt{34}\), \(CP = \sqrt{13}\). Find \(DP\), its distance from \(D\).

\[
\sqrt{53} = 7.280 \quad \text{accept either}
\]

Team
1. As in Event 2D, 24, find that
   \[ y = mx \]
   \[ y - 3 = -m(x + 2) \]
   Intersect at \( x = \frac{3 - 2m}{2m} \), \( y = \frac{3 - 2m}{2} \)
   \( 3 - 2m = 2y \), so \( m = \frac{3 - 2y}{2} \). Then
   \[ x = \frac{3 - 2 \frac{2y}{2} \frac{3 - 2y}{2}}{2} = \frac{2y}{3 - 2y} \]
   Solve for \( y \) to get \( y = \frac{3x}{2x + 2} \).

3. For \( x > -2 \), \( x + 2 = \frac{1}{2} \times + 3 \)
   For \( x < -2 \), \( -(x + 2) = \frac{1}{2} \times + 3 \)
   \[
   \frac{1}{2}x = 1 \\
   x = 2
   \]

5. The equation in standard form is
   \[ 4a(y - n) = (x - m)^2 \]
   \[ y = n + \frac{1}{4a} (x^2 - 2mx + m^2) \]
   Since the coefficient of \( x^2 \) is 1,
   \[ \frac{1}{4a} = 1 \]
   \[ 4a = 1 \]
   and the standard form is \( y - n = (x - m)^2 \).
   When \( y = 0 \), \( x - m = \pm \sqrt{-n} \)
   Since \( n > 0 \), \( -n < 0 \);
   \[ x = m \pm \sqrt{-n} \]

2. Let \( m_1 = AF \), \( m_2 = FB \).
   \( \Delta AEF \sim \Delta ADB \), so \( \frac{c}{b} = \frac{m_1}{m_1 + m_2} \)
   \( \Delta BEF \sim \Delta BCA \), so \( \frac{c}{a} = \frac{m_2}{m_1 + m_2} \)
   Adding, \( \frac{c}{b} + \frac{c}{a} = 1 \); \( c \left( \frac{1}{a} + \frac{1}{b} \right) = 1 \)
   \[ c = \frac{ab}{a + b} = \frac{ab}{b + a} \]

4. In normal form, the line is
   \[ \cos \alpha \times + \sin \alpha \times y = 6 \]
   Intercepts are \( \frac{6}{\cos \alpha} \) and \( \frac{6}{\sin \alpha} \)
   Area = \( \frac{1}{2} \times \frac{6}{\cos \alpha} \times \frac{6}{\sin \alpha} = 60 \)
   \[ \sin 2\alpha = \frac{36}{60} = \frac{3}{5} \]
   \[ \text{Then } \cos 2\alpha = \pm \frac{4}{5} \]
   \[ \sin \alpha = \sqrt{1 - \cos^2 2\alpha} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5} \]
   \[ \cos \alpha = \sqrt{1 + \cos^2 2\alpha} = \sqrt{1 + \frac{16}{25}} = \frac{4}{5} \]
   \[ \frac{3}{\sqrt{10}} + \frac{1}{\sqrt{10}} y = 6 \]
   \[ 3x + y = 6\sqrt{10} \]

6. Draw a line through \( P \), \( 11 AB \), intersecting \( AD \) and \( BC \) at \( M \) and \( Q \), respectively.
   Set \( x = DM \), \( y = MA \).
   \[ MP^2 = DP^2 - x^2 = 74 - y^2 \]
   \[ QP^2 = 13 - x^2 = 34 - y^2 \]
   \[ DP^2 = 40 \]
   \[ DP^2 = 53 \]
   \[ DP = \sqrt{53} \]