1. Express as a single integer the least common multiple of the set \{52, 56, 70\}.

\[
\text{lcm} = 2^3 \cdot 5 \cdot 7 \cdot 13 = 3640
\]

2. Express \[\frac{5 + 3}{7 + 5} = \frac{8}{21}\] as the quotient of two relatively prime integers.

\[
\frac{35}{7} + \frac{5}{18}
\]

3. Find the smallest positive integer \(k\) such that
\[
\frac{7}{39} + \frac{k}{117} = \left(\frac{a}{b}\right)^2
\]
where \(a\) and \(b\) are relatively prime positive integers.

\[
d = 7
\]

4. If \(d\) is the greatest common divisor of 399 and 959, then it is possible to find integers \(r\) and \(s\) so that \(d = 399r + 959s\). Find \(d\), \(r\), and \(s\).

\[
\begin{align*}
\text{lcm} &= 2 \cdot 2 \cdot 13 \\
52 &= 2 \cdot 2 \cdot 13 \\
56 &= 2 \cdot 2 \cdot 7 \\
70 &= 2 \cdot 5 \cdot 7 \\
7 &= 3 \cdot 5 \\
\end{align*}
\]

3. \[
\frac{7}{3} + \frac{1}{3} = \frac{21 + k}{13}
\]
Choose \(k\) so that \(\frac{21 + k}{13} = a^2\)

\[
\begin{array}{c|c|c|c}
  k & \frac{21 + k}{13} & 13 & a^2 \\
  \hline
  5 & 2 & \neq a^2 \\
  18 & 3 & \neq a^2 \\
  31 & 4 & = 2^2 \\
\end{array}
\]

Choose \(k = 31\).

4. \[
\begin{align*}
(959) &= 2(399) + (161) \\
(399) &= 2(161) + (77) \\
(161) &= 2(77) + (7) \\
(77) &= 11(7) \\
d &= 7 \\
7 &= [(959) - 2(399)] - 2[(399) - 2(161)] \\
&= (959) - 4(399) + 4[(959) - 2(399)] \\
&= 5(959) - 12(399) \\
r &= -12, \\
s &= 5
\end{align*}
\]
160° 1. A straight line intersects the x-axis at A, and the y-axis at B as shown in Figure 1, making \( \angle ABO = 70° \). What is the measure of the supplement of \( \angle OAB \)?

140° 2. Referring to Figure 1 and the information given for Problem 1, suppose C is chosen between A and B so that \( OC = BC \). What will be the measure of \( \angle OCA \)?

2 \alpha 3. In right \( \triangle ABC \), let D be the mid-point of the hypotenuse BC, and let \( \alpha \) be the measure of \( \angle BCA \). In terms of \( \alpha \), what is the measure of \( \angle ADB \)?

85° 4. Isosceles \( \triangle ABC \) has its vertex at \( \angle A = 30° \) (Figure 3). A trisector of \( \angle A \) and a trisector of \( \angle B \) meet at R. A trisector of \( \angle B \) and a trisector of \( \angle C \) meet at S. What is the measure of \( \angle BSR \)?

---

Figure 1

From comments below, \( \triangle ABD \) is isosceles with base angles of \( (90-\alpha) \)

\( \therefore \) \( \angle ADB = 180 - 2(90-\alpha) \)

\( \angle ADB = 2\alpha \)

Complete rectangle \( ABCDE \).

Its diagonals intersect at D, making \( AD = CD \), so \( \angle CAD = \alpha \)

---

Figure 4

Base angles of \( \triangle ABC \) are \( \frac{180-30}{2} = 75° \)

\( \therefore \) \( \angle SBC = 25° \)

Complete the figure by drawing the other trisectors of \( \angle A \) and \( \angle C \).

By Morley's Thm, \( \triangle RTS \) is equilateral, making it easy to show \( \angle BSR \neq \angle CST \).

Let \( \alpha = \angle BSR \). Summing angles around S,

\( 130 + \alpha + 60 + \alpha = 360 \)

\( 2\alpha = 170 \); \( \alpha = 85° \)
2006-07 Event 1C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

1. An angle of 195° has a radian measure of \( r\pi \) where \( r \) is a rational number. What is \( r \)?

\[
195° = 180° + 15° = \pi + \frac{1}{3} \cdot \frac{\pi}{4} = \frac{13}{12} \pi
\]

2. The smallest acute angle of a right triangle has a sine of 0.4. In exact terms (not a decimal), what is the sine of the largest acute angle?

3. Given that \( \cos \alpha > \cos \beta > \frac{1}{\sqrt{2}} \), consider the following three statements.

(a) \( \alpha < \beta \) 
(b) \( \alpha > \beta \) 
(c) \( |\alpha| < |\beta| \)

Answer each of the two questions below with as many of a, b, and c as seem correct, or answer none.

Which statements must be true? 
Which statements must be false?

4. Figure 4 shows \( \triangle ABC \) with \( AC = 4 \), \( BC = 2 \), and a perpendicular dropped from \( C \) to \( D \) on \( AB \) so that \( AD = 3DB \). What is the length of \( AB \)?

Let \( \sin \alpha = 0.4 = \frac{3}{5} \)

Then the side opposite \( \beta \) is \( \sqrt{21} \); \( \sin \beta = \frac{\sqrt{21}}{5} \)

In both pictures below, \( \cos \alpha > \cos \beta > \frac{1}{\sqrt{2}} \)

None of the statements must be true or false.
Solutions

Minnesota State High School Mathematics League
Individual Event

2006-07 Event 1D

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

No Calculators in this Event

1. Find the roots of $6x^2 - 13x + 6 = 0$. Both correct answers required for 1 point.

\[
\begin{align*}
\frac{3}{2} & \quad \text{roots: } \frac{2}{3} \\
\end{align*}
\]

2. Write in descending powers of $x$ the equation of a minimal degree polynomial with integer coefficients having $1-i$ and $\frac{1}{2}$ as roots.

\[
2x^3 - 5x^2 + 6x - 2 \quad \text{(Give credit if they include } = 0)\]

3. Write the equation of the horizontal line that will be tangent to the graph of $x^3 - 6x + 2y + 13 = 0$.

\[
y = -2
\]

4. If $p$, $q$, and $r$ are the roots of $x^3 - x^2 + x - 2 = 0$, what is the value of $p^3 + q^3 + r^3$?

\[
\begin{align*}
&1. \quad (2x-3)(3x-2) = 0 \\
&\quad \quad x = \frac{3}{2}, \quad x = \frac{2}{3} \\
&2. \quad (1-i) \text{ and } (1+i) \text{ must be roots} \\
&\quad \quad \frac{(x-(1-i))(x-(1+i))}{2} = \frac{(x-1)^2 + 2}{2} \\
&\quad \quad = x^2 - 2x + 1 + 1 = x^2 - 2x + 2. \\
&\quad \quad \text{Now multiply } (x^2 - 2x + 2)(2x - 1) \\
&\quad \quad = 2x^3 - 5x^2 + 6x - 2. \\
&3. \quad \text{The graph of } \\
&\quad \quad y = -\frac{1}{2} \left( x^2 - 6x + 13 \right) \\
&\quad \quad \text{is at the right.} \\
&\quad \quad \text{To see that the maximum of } y \text{ is } 2, \quad \text{write} \\
&\quad \quad y = -\frac{1}{2} \left( x^2 - 3 \right)^2 \\
&\quad \quad = -2 - \frac{1}{2} (x-3)^2
\end{align*}
\]

4. 

\[
x^3 - x^2 + x - 2 = (x-p)(x-q)(x-r) \\
\quad = x^3 - (p+q+r)x^2 + (pq+pr+qr)x - pqr.
\]

\[
\begin{align*}
&\text{Take from \textbf{[AHSME, 1975]}} \\
&\quad p + q + r = 1 \\
&\quad pq + pr + qr = 1 \\
&\quad pqr = 2 \\
&\quad \text{Since } p, q, \text{ and } r \text{ are roots,} \\
&\quad p^3 - p^2 + p - 2 = 0 \\
&\quad q^3 - q^2 + q - 2 = 0 \\
&\quad r^3 - r^2 + r - 2 = 0 \\
&\quad \text{Adding,} \\
&\quad \text{(*) } p^3 + q^3 + r^3 = (p^2 + q^2 + r^2) \\
&\quad \quad + 2(pq + pr + qr), \\
&\quad \quad 1 = p^2 + q^2 + r^2 + 2(1) \\
&\quad \text{Substitute in (*)}; \\
&\quad p^3 + q^3 + r^3 = -1 + 1 = 6 \\
&\quad p^3 + q^3 + r^3 = 4.
\end{align*}
\]
Minnesota State High School Mathematics League
Team Event

2006-07 Meet 1

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

1. Two mirrors $AB$ and $AC$ are set at $8^\circ$ as in Figure 1. A light source is reflected at $R_1$ where the angle of incidence equals the angle of reflection as indicated. It is then reflected in a similar fashion at $R_2$, $R_3$, etc., until, on the $n$th reflection, it strikes one of the mirrors at a right angle, and then it retraces its path back to $C$. What is the largest possible value of $n$?

2. A newspaper reports that a wall to be built between two warring factions in a city will cost $2$ million per kilometer. Using the fact that a kilometer is $.62$ miles, how much, to the nearest $100,000$, will the wall cost per mile?

3. The right triangle $ADE$ in Figure 3 has a side of length $1$ opposite the $30^\circ$ angle at $A$. From $E$, lines are drawn to $B$ and $C$ on $AD$ making $\angle EBD = 45^\circ$ and $\angle ECD = 60^\circ$. If a line perpendicular to $AD$ erected at $C$ intersects $BE$ at $H$, how long (exact form) is $HF$?

4. Figure 4 shows $\triangle ABC$ with $AC = 4$, $BC = 2$, and a perpendicular dropped from $C$ to $D$ on $AB$ so that $AD = 3DB$. To the nearest tenth of a degree, what is the measure of $\angle ACB$?

5. The graph of $y = \frac{x^2 - x - 4}{2(x - 3)}$ has a vertical asymptote and an asymptote skew to the $x$-axis. Find the area enclosed by the two asymptotes and the $x$-axis.

6. For what choices of $k$ will the graphs of $y = k$ and $y = 2x^3 - 7x^2 - 12x + 6$ have exactly two distinct points of intersection?
Team Event 1 Solutions

1. See Figure 1 on the answer sheet for notation: $\theta = \angle ABR_1$

2. Let $\theta_i$ be the equal angles at $R_i$

3. $\theta_1 = \theta + 8$

4. $\theta_2 = \theta + 8 = \theta + 2(8)$

5. $\theta_3 = \theta + 8 = \theta + 3(8)$

6. $\vdots$

7. $\theta_n = \theta + (n-1)8$

8. $82 = \theta + (n-1)8$ Since $\theta > 0$, the largest choice of $n = 11$.

2. $\frac{\text{cost/mile}}{2,000,000} = \frac{1}{2,000,000} = .0005$

3. $\frac{\text{cost/mile}}{1.613} = 3,226,000 \approx 3.2$ million

4. We use the notation of Figure 3

5. $HC = BC = BD - CD = ED - CD = 1 - \frac{1}{\sqrt{3}} = \frac{3 - \sqrt{3}}{3}$

6. $FC\sqrt{3} = AC = \sqrt{3} \cdot \frac{1}{\sqrt{3}} = 1$ so $FC = 1 - \frac{1}{3} = \frac{2}{3}$

7. $HF = \frac{2}{3} - \frac{3 - \sqrt{3}}{3} = \sqrt{3} - 1$

4. As in Problem 4 of Event C, show that $DB = \frac{\sqrt{6}}{2}$. Then $AD = \frac{3\sqrt{6}}{2}$

5. $\cos(\angle CAD) = \frac{3\sqrt{6}}{4} = \frac{3\sqrt{6}}{4}$ so $\angle CAD = 23.3^\circ$

6. $\cos(\angle CBD) = \frac{\sqrt{6}}{2} = \sqrt{\frac{6}{4}} = \sqrt{\frac{3}{2}}$ so $\angle CBD = 52.2^\circ$

7. $\therefore \angle ACB = 180^\circ - (23.3 + 52.2) = 104.5^\circ$

5. By long division, $y = \frac{x}{2} + 1 + \frac{2}{2x - 6}$

As $x \to \infty$, $y \approx \frac{1}{2}x + 1$

The shaded area bounded by $y = \frac{1}{2}x + 1$, $x = 3$, and the $x$-axis has area $= \frac{1}{2} (\frac{5}{2}) (5)$

6. We seek $k$, for which $2x^3 - 7x^2 - 12x + 6 = k$

has double roots.

By synthetic division: