Minnesota State High School Mathematics League
Individual Event

2006-07 Event 4A

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

1. Express as the quotient of two relatively prime integers
   \[
   4^2 + 4^2 + 4^1 + 4^0 + 4^{-1} + 4^{-2}
   \]

2. Simplify \(\frac{(a + a^{-1})^{-1}}{a(a^2 - 1)^{-1}}\), leaving an expression in which any exponents that appear are positive.

3. Given \(f(x) = \frac{x - 1}{x + 1}\), find \([f(x^{-1})][f(x)]^{-1}\)

4. Find all lattice points interior to the first quadrant, that is all points \((m, n)\) where \(m\) and \(n\) are positive integers, that lie on the graph of
   \[x^2 + y^2 + 2xy - 4x - 4y - 5 = 0\]

Name ___________________________ Team ___________________________
2006-07 Event 4B

All questions in this event refer to the figure at the right in which A and B are placed on a circle of radius 5 so that $\angle XOA = 45^\circ$ and $\angle XOB = 135^\circ$. P is allowed to move along the circle so that $\alpha = \angle XOP$ varies from $0^\circ$ to $360^\circ$. Let $\beta = \angle APB$, (so $\beta$ is undefined when $P$ coincides with either $A$ or $B$).

1. What is the length of the arc moving from $A$ counterclockwise to $Z$?

2. What is the area of the region bounded by the arc described in Problem 1 and the two radii $OA$ and $OZ$?

3. On the axes provided in Figure 3, graph $\beta$ as a function of $\alpha$ as $\alpha$ varies from $0^\circ$ to $360^\circ$. Use an open circle to show "gaps" in the graph at the points where $\beta$ is undefined.

4. In the given figure, suppose that with $A$ still fixed, we now allow $B$ to move on the circle so that $\theta = \angle BOA$ varies. Extend $BP$ to $M$ so that $AP = PM$. What is the measure of $\omega = \angle AMP$ in terms of $\theta$?

Figure 3

Name ___________________________ Team ___________________________
2006-07 Event 4C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

NO CALCULATORS IN THIS EVENT

1. The sequence \( \{a_n\} \) is defined recursively by
   \[ a_0 = 2, \ a_1 = 5, \ \text{and} \ a_{n+2} = a_n + a_{n+1} \text{ for } n = 0,1,2,... \]
   Find \( a_5 \).

2. The expansion of \( \left( \frac{2}{a} + \frac{a^2}{4} \right)^8 \) includes a term of the form \( rm \) where \( r \) is an integer.
   What is \( r \)?

3. A function \( f \) is said to be odd if \( f(x) = -f(-x) \) for all \( x \). Indicate in the blank at the left (using i, ii, iii) which of the following functions are odd.
   \[ (i) \ f(x) = -(x-2) + \frac{1}{3}(x-2)^3 \]
   \[ (ii) \ f(x) = x \cos x \]
   \[ (iii) \ f(x) = x \sin x \]

4. It is well known that the sum \( 1 + 2 + 3 + \ldots + n \) can be expressed in the closed form \( \frac{n(n+1)}{2} \). Find a similar closed form for the sums
   \[ (a) \ 1^2 - 2^2 + 3^2 - 4^2 + \ldots - n^2 \text{ when } n \text{ is even} \]
   \[ (b) \ 1^2 - 2^2 + 3^2 - 4^2 + \ldots + n^2 \text{ when } n \text{ is odd} \]

Name _______________________________ Team __________________________
All questions refer to the line L with equation \( x - y + 4 = 0 \) and the point \( P(-1,1) \) shown in the figure at the right.

1. What is the distance from the point \( P \) to the line \( L \)?

2. Points equidistant from \( P \) and the line \( L \) lie on a parabola. Give the coordinates of the vertex of this parabola.

3. Points equidistant from \( P \) and the line \( L \) lie on a parabola. Where does this parabola cross the y-axis?

4. Write in the form \( ax^2 + bxy + cy^2 + dx + ey + f = 0 \) the equation of the parabola described in problems 2 and 3.

Name_________________________________________ Team____________________________________
1. How many lattice points are there that are interior to the first quadrant, that is all points \((m,n)\) where \(m\) and \(n\) are positive integers, that lie on the graph of 
\[x^2 + y^2 + 2xy - kx - ky - k - 1 = 0\]

2. Use inequalities to order from the smallest to the largest the numbers 
\[5^{(s!)} ,\  (5!)! ,\  (5^3)!\]

3. For \(n > 4\), \(f(x) = x^n - n(x-1) - 1\) can be written in the form \(f(x) = (x-1)^2 g(x)\). Find, in terms of \(n\), \(g(1)\).

4. Let \(x = \frac{1}{b-c}\), \(y = \frac{1}{c-a}\), \(z = \frac{1}{a-b}\). Then \(x^2 + y^2 + z^2 = [f(x,y,z)]^2\) where \(f(x,y,z)\) does not involve \(a\), \(b\), or \(c\). Find \(f(x,y,z)\) in simplified form.

5. A function \(f\) is said to be \textit{odd-like} for \(a \leq x \leq b\) if \(f(a + b - x) = -f(x)\). Indicate in the blank at the left (using i, ii, iii) which of the following functions are odd-like.

   (i) \(f(x) = -(x-2) + \frac{1}{3}(x-2)^3\) on \(1,3\)

   (ii) \(f(x) = x \cos x\) on \(0,\pi\)

   (iii) \(f(x) = x \sin x\) on \(0,\pi\)

6. The large circle in Figure 6 is tangent to both coordinate axes and the circle \(x^2 + y^2 = 1\). The line joining the centers of the circles intersects the small circle at \(A\) and \(B\), and \(CD\) is a diameter of the large circle that is perpendicular to the x-axis. Find the length of \(BD\).