The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

1. Express \( \frac{x^4 - xy^3 + x^3y - y^4}{x^2 - y^2} \) as a polynomial in \( x \) and \( y \).

2. Jane is half as old as Terry will be in six years when the sum of their ages will be 75. How old is Jane now?

3. Solve the system

\[
\begin{align*}
x + 6y + 5z &= 3 \\
y - z &= 2
\end{align*}
\]

4. Find the largest value of \( s \) and the smallest value of \( t \) such that for all \( x \) satisfying \(-6 < -2x < 8\) and for all \( y \) satisfying \(1 < |y - 4| < 3\),

\[
\begin{align*}
(a) & \quad s = -28 \quad t = 21 \\
(b) & \quad s = -4 \quad t = 3
\end{align*}
\]
1. Figure 1 shows a square $ABCD$ with sides of length 4. Concentric circles, one inscribed in the square, the other circumscribed about it, bound a washer shaped region. What is the area of this region?

$4\pi$

2. The difference between the largest and smallest angles of a parallelogram is 66°. What is the measure of the smallest angle?

$57°$

3. Figure 3 shows a square $ABCD$ with sides of length 4, and four inscribed semicircles centered at $E$, $F$, $G$, and $H$. These semicircles intersect at $K$ and overlap to form four (shaded) leaves. What is the total area of the four shaded leaves?

$8\pi - 16$

4. Figure 4 shows four points connected by six lines, three short ones each having length 1, and three others each having the same length $x > 1$. Find $x$, accurate to three places to the right of the decimal.

1.618

(Refer to the figure at the right)

1. Radius of the large circle is $KC = R, R^2 = 4 + 4 = 8$. Radius of the small circle is $r = 2$. The enclosed area $= \pi R^2 - \pi r^2 = 4\pi$.

2. Let $\alpha = \text{small } \angle$, $B = \text{large } \angle$.

$\begin{align*}
\alpha & = 66 \\
B + \alpha & = 180 \\
2B & = 246 \\
\alpha & = 57^\circ
\end{align*}$

3. Focus on square $DEKH$

Cross hatch area $= 2 - \frac{1}{4} \pi^2$

Leaf area $= 4 - 2 \text{ (Cross hatch area)}$

$= 4 - 2 (4 - \pi) = 2\pi - 4$

Area of 4 leaves $= 4 (2\pi - 4)$

From $\triangle MNQ$, $(1 + t)^2 + h^2 = (1 + 2t)^2$

From $\triangle NPQ$, $h^2 + t^2 = 1; h^2 = 1 - t^2$

Substitute,

$(1 + t)^2 + (1 - t^2) = (1 + 2t)^2$

Simplify, $4t^2 + 2t - 1$

$t = \frac{-1 \pm \sqrt{5}}{4}$. Use $t = \frac{\sqrt{5} - 1}{4} \approx 0.30902$
1. Let $\alpha$ and $\beta$ be two angles, $0 < \alpha < \beta < 2\pi$, such that $\sin \alpha = \sin \beta = -\frac{1}{2}$. What is the radian measure of $\beta - \alpha$?

\[
\frac{2\pi}{3}
\]

2. Figure 2 shows an equilateral triangle inscribed in a right isosceles triangle having legs of length 4. Give the length of the sides of the equilateral triangle in exact, rationalized form.

\[
\frac{4\sqrt{6}}{3}
\]

3. A cooler contains 4 bottles of carbonated mineral water and 6 bottles of non-carbonated spring water. Not realizing there is a choice, Alicia and Beth each plunge their hands into the ice and grab a bottle. What is the probability that both girls get the same kind of water?

\[
\frac{7}{15}
\]

4. A complex number $a + bi$ lies on the unit circle if $a^2 + b^2 = 1$. Express each of the two roots of $x^2 = 4 + 3i$ in the form $r(a + bi)$ where the real numbers $r$, $a$, and $b$ are expressed in exact form and $a + bi$ lies on the unit circle.

\[
\frac{\sqrt{5}}{2} \left( \frac{3}{\sqrt{10}} + \frac{i}{\sqrt{10}} \right)
\]

or

\[
\frac{\sqrt{5}}{2} \left( \frac{3}{\sqrt{10}} - \frac{i}{\sqrt{10}} \right)
\]

1. $\beta - \alpha = \frac{11\pi}{6} - \frac{7\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$

2. Apply the law of sines to $\triangle ABC$.

\[
\frac{x}{\sin 45^\circ} = \frac{4}{\sin 120^\circ}
\]

\[
x = \frac{4 \cdot \frac{\sqrt{3}}{2}}{\sqrt{2}} = \frac{4\sqrt{3}}{3}
\]

3. $\triangle ABC$.

\[
3^\circ \rightarrow M \left( \frac{6}{45}, \frac{6}{45} \right) \text{ both mineral}
\]

\[
4^\circ \rightarrow M \left( \frac{6}{45}, \frac{6}{45} \right)
\]

\[
\frac{1}{2} \rightarrow \left( \frac{12}{45}, \frac{15}{45} \right) = \frac{7}{15}
\]

4. $(4 + 3i) = 5 \left( \frac{4}{5} + \frac{3}{5}i \right) = 5 (\cos \alpha + i \sin \alpha)$

where $\alpha = \cos^{-1} \frac{4}{5}$.

$(4 + 3i) = \pm 5 \left( \cos \frac{\alpha}{2} + i \sin \frac{\alpha}{2} \right)$

$\cos \frac{\alpha}{2} = \sqrt{\frac{1 + 4/5}{2}} = \frac{3}{\sqrt{10}}$; $\sin \frac{\alpha}{2} = \sqrt{1 - 4/5} = \frac{1}{\sqrt{10}}$
2006-07 Tournament Event D

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THIS IS A NO CALCULATOR EVENT

1. Express $4^{31} - (4^3)^2$ as a single integer.

2. Point $A$ is in the first quadrant, located so that with $O$ designating the origin, $OA = 2$ and $OA$ makes an angle of $30^\circ$ with the positive $x$-axis. Write an equation of the line through $A$, perpendicular to $OA$.

   \[ y - 1 = -\sqrt{3} (x - \sqrt{3}) \quad \text{or} \quad \sqrt{3} x + y = 4 \]

3. The quadratic equation $x^2 + bx + c = 0$ has roots $1 + m + \sqrt{1 + m^2}$ and $\frac{2m}{1 + m + \sqrt{1 + m^2}}$. Express $b$ as a linear expression in $m$.

   \[ b = -2 - 2m \]

4. Using common logarithms (that is, logarithms to the base 10), $\log 5 = 0.69897$. Find $\log \sqrt{\frac{10}{5}}$ rounded correct to three places to the right of the decimal.

   \[ \log \sqrt{\frac{10}{5}} = \log 2 \sqrt{2} = \log 2 + \frac{1}{2} \log 2 \]

\[ \sqrt{\frac{10}{5}} = \frac{2}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{2}} \]

\[ \log \frac{\sqrt{5}}{\sqrt{2}} = \log \sqrt{5} \cdot \log \frac{\sqrt{2}}{\sqrt{5}} \]

\[ \log \sqrt{5} = \frac{1}{2} \log 5 = 0.30103 \]

\[ \log \sqrt{2} = \frac{1}{2} \log 2 = 0.15052 \]

\[ \log \sqrt{\frac{10}{5}} = \log 2 \sqrt{2} = \log 2 + \frac{1}{2} \log 2 = 0.69897 \]
1. The parallelogram $ABCD$ with $EF \parallel AB$ and $\angle BAD = 60^\circ$ (Figure 1) is often used to create an optical illusion. The viewer is asked which is longer, $BE$ or $EC$. Suppose $AB = 1$ and $BC = 2$. How long should $AE$ be so that $BE = EC$?

2. The polynomial $p(x)$ has non-negative coefficients; $p(1) = 12$ and $p(20) = 18,081$. Find $p(x)$.

3. Write equations for all lines passing through $(1,-1)$ having an $x$-intercept that is twice the slope of the line. OR $x + 2y + 1 = 0$

4. The mean of the set of real numbers $\{x_1, x_2, \ldots, x_9\}$ is $k$. The mean of the set $\{x_1, x_2, \ldots, x_9, 17\}$ is $8$. What is $k$?

5. $P$ is a point in the interior of $\triangle ABC$ that has $AB = 13$, $BC = 7$, and $CA = 8$. The distance from $P$ to the side $AB$ is $\frac{\sqrt{3}}{4}$; its distance to the side $BC$ is $\frac{5\sqrt{3}}{4}$. Find the distance from $P$ to the side $AC$.

6. Here is a system of three equations in the three unknowns $x, y, z$.

The unknown $x$ can be written as a single term involving only of $a, b, c$. Find this term.
1. Since $BF \parallel AE$, $\angle AEB = \alpha$

Since $\triangle BEC$ is isosceles, $\angle BCE = \alpha$

Apply the law of cosines to $\triangle ABE$

$$d^2 = b^2 + l - 2b \cos 60^\circ = b^2 - b$$

Apply the law of cosines to $\triangle CDE$

$$d^2 = a^2 + 1 - 2a \cos 120^\circ = a^2 + 1$$

\[ b^2 + l - b = a^2 + 1 + a \]

\[ b^2 = a^2 + (a+b) = (2-b)^2 + 2 \]

Solve $b^2 = 4 - 4b + b^2 + 2j$; $b = \frac{3}{2}$

(The figure above, drawn to size, illustrates the optical illusion)

2. Since $p(t) = \text{sum of all coefficients}$, all coefficients $\leq 12$

Write $18081$ using base 20.

$18081 = 2(20) + 5(20) + 4(20) + 1$

\[ p(x) = 2x^3 + 5x^2 + 4x + 1 \]

3. 

There appear to be two such lines.

Such a line will have equation

$$y + 1 = m(x - 1)$$

and its $y$ intercept is found by setting $y = 0$. Solving for $x$, $x = \frac{1 + m}{m}$

We seek $m$ so that $\frac{1 + m}{m} = 2m$. Solve $2m^2 - m - 1 = 0$; $m = -\frac{1}{2}$ or 1

4. \[ \frac{x_1 + x_2 + \cdots + x_9 + 17}{10} = 8 \]

We know \[ x_1 + x_2 + \cdots + x_9 = 9k \]

\[ 9k + 17 = 8(8); \quad 9k = 63; \quad k = 7 \]

5. drawing is to scale; $\frac{1}{4} = 1$

Using Heron's formula, $s = \frac{13 + 7 + 8}{2} = 14$

Area $\triangle ABC = \sqrt{14(1)(7)(6)} = 14\sqrt{3}$

Area $\triangle ABP = \frac{1}{2}(3)\frac{\sqrt{3}}{4} = \frac{13\sqrt{3}}{8}$

Area $\triangle ABP = \frac{1}{2}(7)\frac{5\sqrt{3}}{4} = \frac{35\sqrt{3}}{8}$

Area $\triangle ABP = \frac{1}{2}(8)x = 4x$

\[ 14\sqrt{3} = 4\frac{\sqrt{3}}{4} + 4x \]

\[ x = 2\sqrt{3} \]

6. There are many ways to solve this system. An elegant way is to consider

(i) $t^3 - 2t^2 - yt - x = 0$

This equation has roots of $a, b,$ and $c$, so it can be written in the form

$$(t - a)(t - b)(t - c) = 0$$

Multiply this out;

(ii) $t^3 - (a+b+c)t^2 + (ab+ac+bc)t - abc = 0$

Compare (i) and (ii); $-x = -abc$