1. Given that \((y + 7) + (2y + x^2) = (3 + x^2) + 3\), find the value of \(y\).

\[
y = \frac{-1}{3} \text{ or } -0.333.
\]

\[
3y + 7 + x^2 = 6 + x^2 \Rightarrow 3y = -1, \text{ etc.}
\]

2. The Sir Charge car rental company charges $45 per day to rent a car, plus $6.00 per gallon of gas used. The car can drive 26 miles per gallon. If Teddy rents this car from Sir Charge for 5 days and drives it a total of \(m\) miles, write a linear expression in terms of \(m\) which describes Teddy’s total cost.

\[
\frac{225 + \frac{3}{13}m}{5 \text{ days}} = \frac{225}{5} \text{ miles} \times \frac{1 \text{ gallon}}{26 \text{ miles}} \times \frac{6 \text{ gallon}}{3 \text{ miles}} = \frac{3}{13}m.
\]

3. Don’s calculator has some warped circuits and does not input integers correctly. When Don enters an odd integer \(n\), the calculator interprets it as \(n + 3\). When Don enters an even integer \(m\), the calculator interprets it as \(m/2\). If Don entered the numbers 15, 24, 33, and \(x\), and the calculator claimed the sum of those numbers was 101, find \(x\).

If \(x\) is odd, then the sum is \((15 + 3) + (24/2) + (33 + 3) + (x + 3) = 101\).

Solving, \(x = 32\), which contradicts the fact that \(x\) is odd.

If \(x\) is even, then the sum is \((15 + 3) + (24/2) + (33 + 3) + (x /2) = 101\).

Solving here yields \(x = 70\).

4. Ole, who occasionally lies, wrote down an integer \(N\), and then gave his friend Lena the following inequalities:

- A. \(N - 13 < 50 - N\)
- B. \(N + 8 < 100 - 2N\)
- C. \(3N + 60 < 5N + 3\)
- D. \(5N - 20 < N + 99\)

As it turned out, exactly two of those statements were lies. Find the value of \(N\).

\[
N = 31.
\]

A is true, since otherwise it would make both B and D false.

C is true, since no other statements can be false with it. So \(30\% < N < 31.5\).
2008-09 Event 2B
SOLUTIONS

1. The lengths of the sides of a scalene triangle, listed in size order, are 5, \( x \), and 15. How many possible values for \( x \) are there, given that \( x \) is an integer?

There are 4 possible values. By the Triangle Inequality, \( 5 + x > 15 \Rightarrow x > 10 \). Since the sides are in size order, \( x = 11, 12, 13, \) or 14.

2. Point \( P \) is chosen along leg \( BC \) of right triangle \( ABC \) so that \( BP = PA \) (Figure 2). If \( BC = 10 \) and \( AC = 4 \), find \( BP \).

\[
BP = \frac{29}{5} \text{ or } \frac{4}{5} \text{ or } 5.8.
\]

3. In an isosceles triangle with a 30° vertex angle, the perpendicular bisector of one leg divides the other leg into the ratio \( k : 2 \). If \( k \geq 2 \), find \( k \).

\[
k = \frac{2}{\sqrt{3} - 1} \text{ or } \sqrt{3} + 1 \text{ or } 2.732.
\]

4. In triangle \( ABC \), \( m\angle ABC = m\angle ACB = 72^\circ \). Find the ratio \( AB : BC \), and express it as a decimal accurate to three decimal places.

\[
AB : BC = \frac{1.618}{1} \text{ or } \left(1 + \sqrt{5}\right) : 2 \text{ or } 2 : \left(\sqrt{5} - 1\right)
\]

Labeling angles, \( \alpha = 36^\circ \), \( 2\alpha = 72^\circ \), etc.
Also, let \( BF = FC = BD = x \), and \( DF = y \). Set \( \frac{\alpha}{\alpha} = r \).
\( \triangle CDB \sim \triangle ABC \Rightarrow \frac{CD}{DB} = \frac{x+y}{x} = 1 + \frac{1}{r} = \frac{\alpha}{\alpha} = r \).
Use a calculator’s equation solver on \( 1 + \frac{1}{r} = r \).
Questions 1 and 2 refer to Figure 1 in which \( \tan \alpha = \frac{5}{12} \), \( \tan \beta = 4\sqrt{5} \).

1. Find \( \sin 2\alpha \).

\[
\sin 2\alpha = 2 \sin \alpha \cos \alpha = 2 \left( \frac{5}{13} \right) \left( \frac{12}{13} \right)
\]

or \( \frac{120}{169} \) or \( .710 \).

2. Find \( \cos 2\beta \).

\[
\cos 2\beta = \pm \sqrt{\frac{1 + \cos 2\beta}{2}} = \pm \sqrt{\frac{1 - \frac{1}{6}}{2}} = \pm \sqrt{\frac{4}{9}} = \pm \frac{2}{3}
\]

or \( \frac{-2}{3} \) or \( -.667 \).

3. In Figure 3, \( \theta_1 + \theta_2 = 45^\circ \). Find BC.

\[
\tan \theta_2 = \tan (45^\circ - \theta_1) = \frac{\tan 45^\circ - \tan \theta_1}{1 + (\tan 45^\circ)(\tan \theta_1)}
\]

\[
= \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}} = \frac{2}{4} = \frac{1}{2}
\]

\[
x = \sqrt{10} \cdot \tan \theta_2 = \sqrt{10} \cdot \frac{1}{2}
\]

or \( \frac{\sqrt{10}}{2} \) or \( 1.581 \).

4. \( ABCD \) is a kite-shaped quadrilateral (Figure 4) with \( BC = CD \) and \( AB = AD = x \).

\( m\angleBAD = 45^\circ \); \( m\angleBCD = 135^\circ \). From D, drop a line perpendicular to \( AB \), meeting \( AC \) at E, and \( AB \) at F.

Find \( EF \) in terms of \( x \).

\[
\frac{\tan 45^\circ}{2} = \frac{BC}{x} = \frac{1 - \cos 45^\circ}{\sin 45^\circ} = \frac{1 - (\sqrt{2}/2)}{(\sqrt{2}/2)} = \frac{2 - \sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1
\]

\[
\sin 45^\circ = \frac{EF}{x} \Rightarrow FD = x \cdot \frac{\sqrt{2}}{2} = AF
\]

\[
\frac{AF}{AB} = \frac{EF}{BC} \Rightarrow \frac{\sqrt{2}}{2} = \frac{EF}{x(\sqrt{2} - 1)} \Rightarrow EF = x \cdot (\sqrt{2} - 1) \cdot \frac{\sqrt{2}}{2}
\]

e tc.
1. Write the equation of the line which passes through the origin and is parallel to \(2x - 3y = 7\).

\[
y = \frac{2}{3}x \quad \text{or} \quad 2x - 3y = 0.
\]

\(2x - 3y = 7\) has slope \(\frac{-A}{B} = \frac{-2}{3}\), and any line through the origin has \(y\)-intercept 0.

2. Find the intersection point of the lines \(y = \frac{7}{5}x - 10\) and \(y = \frac{3}{4}x\).

\((x, y) = \left(\frac{200}{13}, \frac{150}{13}\right)\) or \((15.385, 11.538)\).  \[\frac{3}{4}x = \frac{7}{5}x - 10 \Rightarrow \left(\frac{15}{20} - \frac{28}{20}\right)x = -10 \Rightarrow x = \frac{200}{13} \Rightarrow y = \frac{3}{4}x = \frac{3}{4} \cdot \frac{200}{13} = \frac{150}{13}\]

Graders: deduct 1 point for omitting \(\pm\), or for placing \(\pm\) on the \(y\)-coordinate.

(If a student makes both mistakes, give no credit.)

3. Find all ordered pairs \((x, y)\) that satisfy

\[
\begin{align*}
x^2 + y^2 &= 5 \\
x^2 + (y + 4)^2 &= 6^2
\end{align*}
\]

\((x, y) = \left(\pm \frac{\sqrt{95}}{8}, \frac{15}{8}\right)\).  Subtracting eqn. 1 from eqn. 2 yields \(y^2 + 8y + 16 - y^2 = 36 - 5\) \(8y = 15 \Rightarrow y = \frac{15}{8}\) \(x^2 + \left(\frac{15}{8}\right)^2 = x^2 + \frac{225}{64} = 5 \Rightarrow x^2 = \frac{95}{64}\)

4. Two lines intersect in the \(xy\)-plane. The first line has \(x\)-intercept \(2p\) and \(y\)-intercept \(2p\), while the second line has \(x\)-intercept \(p\) and \(y\)-intercept \(3p\). If their intersection point is concurrent with the line \(x = 3\), find \(p\).  (Credit: 1998 NC HS Math Contest)

\[
p = 6.
\]

\[\begin{align*}
\frac{x}{2p} + \frac{y}{2p} &= 1 \Rightarrow x + y = 2p \\
\frac{x}{p} + \frac{y}{3p} &= 1 \Rightarrow 3x + y = 3p
\end{align*}\]
1. If $a$, $b$, and $c$ are distinct in the system

\[
\begin{align*}
  a^3 + 3a &= -14 \\
  b^3 + 3b &= -14 \\
  c^3 + 3c &= -14
\end{align*}
\]

find the value of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

$-\frac{3}{14}$ or $-0.214$.

2. The altitude and the median from vertex $A$ of triangle $ABC$ are 4 and 5 units long, respectively. The altitude bisects the angle determined by side $AB$ and the median. Find $AC$.

$\sqrt{97}$ or $9.849$.

3. Referring to problem #3 from Event A, Don bought a new calculator to replace his old one. Sadly, this new calculator also inputs integers incorrectly. It interprets the odd integer $n$ as $4n$, and the even integer $m$ as $m + 5$. When Don used the new calculator to multiply the positive integers $a$ and $b$, the result was 4444. Find the smallest possible value that the calculator would return for the sum $a + b$.

145.

4. Susan, who had no calculator available, was asked to solve $\sqrt{6} \cos x + \sqrt{2} \sin x = 2$. She squared both sides and simplified to get the equation $2 \cos^2 x + b \sin 2x = 1$. In exact form, what is $b$?

$2\sqrt{3}$. **Graders:** Note that the problem requires this answer in exact form.

5. Points $A$ and $B$ are on the same side of line $m$ and are 5 and 7 units away from $m$, respectively. $A$ and $B$ are 17 units apart. For all points $P$ on line $m$, what is the smallest possible value of $AP + BP$?

$\sqrt{429}$ or $20.712$.

6. In a triangle, the lengths of the three medians are 9, 12, and 15. Find the length of the side to which the longest median is drawn.

10.
1. \( a, b, \) and \( c \) are actually the three roots of \( x^2 + 3x + 14 = 0 \).

\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab + ac + bc}{abc} = \frac{-3}{14}.
\]
(This last step uses Vieta’s Formulas, which relate the coefficients of a polynomial to its roots. For more info: http://www.artofproblemsolving.com/Resources/Papers/PolynomialsAK.pdf)

2. \[
\begin{align*}
\text{Drawing a good picture can make problems easy:}
\text{Triangles AED and AEB are both 3-4-5, and since }
\overrightarrow{AD} \text{ is a median, BD = CD = 6.}
\text{Applying the Pyth. Thm. to right triangle AEC,}
\end{align*}
\]

\[
x^2 = 4^2 + 9^2 = 97 \quad \Rightarrow \quad x = \sqrt{97}.
\]

3. Don’s calculator turns even numbers into odds, and odds into multiples of 4. So, for a product to equal \( 4444 = 4 \cdot 1111 \), one of the original numbers must be odd (to become a multiple of 4), while the other must be even (to become odd by adding 5). Suppose \( a \) is odd and \( b \) is even. Then \( 4a(b+5) = 4444 \Rightarrow a(b+5) = 1111 \). Either \( 1111 = 11 \cdot 101 \), or \( 1111 = 1 \cdot 1111 \).

Since \( b \) is positive, \( b + 5 \) can only equal 11, 101, or 1111, yielding \( b = 6, 96, \) or \( 106 \).

This forces \( a = 101, 11, \) or 1, respectively. Don’s calculator will output \( a + b \) as \((4a) + (b + 5)\).

Checking all three possible pairs of integers \((a, b)\), the smallest sum is \((4 \cdot 11) + (96 + 5) = 145\).

4. \[
\left( \sqrt{6} \cos x + \sqrt{2} \sin x \right)^2 = (2)^2
\]

\[
6 \cos^2 x + 2 \sqrt{12} \sin x \cos x + 2 \sin^2 x = 4
\]

\[
4 \cos^2 x + \left( 2 \cos^2 x + 2 \sin^2 x \right) + 2 \sqrt{3} \left( 2 \sin x \cos x \right) = 4
\]

5. \[
\begin{align*}
\overrightarrow{AP} + \overrightarrow{PB} &= \overrightarrow{AP} + \overrightarrow{PB'} \\
&= \sqrt{B'Q^2 + QA^2} \\
&= \sqrt{12^2 + (AB^2 - BQ^2)} \\
&= \sqrt{144 + (17^2 - 2^2)} \\
&= \sqrt{429}
\end{align*}
\]

6. In the diagram, median \( \overrightarrow{BE} \) is extended to point \( P \) so that \( ME = EP \). The lengths of \( AM, ME, \) and \( MC \) are all determined by the fact that the centroid divides each median into segments of length ratio 2:1. Since \( AC \) and \( MP \) bisect each other, \( AMCP \) is a parallelogram.

Further, \( AMP \) is a 6-8-10 triangle, so \( \angle MAP = 90^\circ \), and \( AMCP \) is a rectangle! \( AC = MP = 10 \).