2008-09 Event 4A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

1. If \( f(x) = x^{\frac{2}{3}} + x^{\frac{3}{2}} \), find \( f(64) \).

\[ f(64) = \] 

2. Simplify the expression \( \frac{\sqrt{x}}{x^{\frac{1}{2}} + x^{-\frac{1}{2}}} \) so that the numerator and denominator are both linear expressions in \( x \).

\[ \text{____________________________} \]

3. If \( \frac{2a+b}{a+2b+c} = \frac{1}{2} \) and \( \frac{c-b}{c-b+a} = \frac{1}{3} \), determine the ratio \( a : b : c \), where \( a, b, \) and \( c \) are positive integers and their greatest common factor is 1.

\[ a : b : c = \] 

4. \( g(x) \) satisfies the equation \( \left( \sqrt{a-a} + \frac{1}{a} \right) \left( \sqrt{a+a} - \frac{1}{a} \right) = a - g \left( a^2 \right) - g \left( \frac{1}{a^2} \right) \).

If \( g(3) = 2 \), compute \( g(5) \).

\( g(5) = \) 

Name _______________________________ Team _______________________________
Minnesota State High School
Mathematics League
Individual Event

2008-09 Event 4B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

1. In Figure 1, $BC$ is a diameter of circle $A$, and $DE$ is a tangent. If $m\overarc{CE} = 2 \cdot m\overarc{BE}$, find $m\angle D$.

$m\angle D = \underline{\phantom{0000000000}}$

2. In Figure 2, a snaking series of segments of length 1 moves “east”, then “north”, then “northeast” from the origin. Circles $P$ and $Q$, with radii of lengths $p$ and $q$ respectively, are inscribed in two isosceles right triangles, each of which have a leg lying on an axis, and a hypotenuse lying on the line $y = x$. Find the ratio $p/q$.

$p/q = \underline{\phantom{0000000000}}$

3. In Figure 3, $MN$ is a diameter, points $V, W, X, Y, Z$ divide the semicircle above $MN$ into congruent arcs, and points $A, B, C$ divide the semicircle below $MN$ into equal arcs. Find the sum of the angles at the 5 points of the shaded star ($A, C, N, K, M$).

____________________________

4. In Figure 4, diameter $AB = 2$, $PB$ and $PD$ are secant lines, and $D$ is the midpoint of $ADB$. If $CD = 1$, find the length of chord $AC$.

$AC = \underline{\phantom{0000000000}}$

Name ____________________________
Team ____________________________
Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

1. If $a$ and $b$ are distinct whole numbers, and $a! = b! = x$, find $x$.

   $x = \underline{\hspace{2cm}}$

2. The third term of an arithmetic sequence is 1, the ninth term is 25, and the $n$th term is 81. What is the value of $n$?

   $n = \underline{\hspace{2cm}}$

3. The sequence $\{c_n\}$ is defined by $c_1 = 1$, and for $n > 1$, $c_n$ is the smallest integer satisfying $c_n \cdot c_{n-1} \geq n$. Evaluate the expression $\frac{c_1 + c_2 + c_3 + \ldots + c_{2008}}{2008}$.

   _______________________

4. For certain ordered pairs of integers $(x,n)$, with $n > 0$,

   $\sum_{k=0}^{n} x^k 3^{n-k} \binom{n}{k} = 4096$.

   Compute the sum of all possible values of $x$ from these ordered pairs.

   _______________________

Name  ____________________________  Team  ____________________________
2008-09 Event 4D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

1. Identify the type of conic described by the equation \(4x^2 + y^2 - 8x - 2y = -1\).

   __________________________

2. An ellipse with center (4, 2) and focus (7, 2) is tangent to the y-axis. Find the length of the ellipse’s minor axis.

   __________________________

3. Points \(P\) and \(Q\) on the graph of \(13y = 3x^2 - 30\) are the centers of circles, both of which are tangent to both axes. These circles are tangent to each other at the point (0, \(k\)). Find both possible values of \(k\).

   \(k = \) __________________________

4. A certain parabola passes through the points (5, 1) and (13, -7) and has the y-axis as its directrix. Find the coordinates of all points at which the vertex of this parabola could be located.

   __________________________

Name ___________________________ Team ___________________________
Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

1. The distance from the point \((x_0, y_0)\) to the line \(Ax + By + C = 0\) is given by \(\frac{Ax_0 + By_0 + C}{\pm \sqrt{A^2 + B^2}}\), where the sign in the denominator is always opposite the sign of \(C\).

Find the equation of the line bisecting the acute angle formed by \(3x - 4y - 15 = 0\) and \(5x + 12y - 13 = 0\). (Hint: an angle bisector’s points are equidistant from the sides of the angle.)

2. Recall Figure 2 from Event B.

The simplest expression for \(PQ\), the distance from the center of circle \(P\) to the center of circle \(Q\), is of the form \(\frac{\sqrt{a + b^2}}{2}\).

\[ PQ = \quad \text{[2 points]} \text{ Find } PQ \text{ accurate to 3 decimal places.} \]

\[ (a, b) = \quad \text{[2 points]} \text{ Find } (a, b). \]

3. The sequence \(\{c_n\}\) is defined by \(c_1 = 1\), and for \(n > 1\), \(c_n\) is the smallest integer satisfying \(c_n \cdot \max\{c_1, c_2, \ldots, c_{n-1}\} \geq n\). Compute the smallest integer \(k\) such that \(c_k = c_{2008}\).

\[ k = \]  

4. \(f(x)\) and \(g(x)\) are linear functions. If the graph of \(f[g(3x) + 2x]\) has slope 9, and the graph of \(g[2f(4x) + 1]\) has slope 20, find the slope of \(f(x)\).

\[ \]  

5. A 70-foot-wide rectangular (box-shaped) barge is steered through a perfectly centered bridge with the following sign posted: “Maximum height 30 ft. Maximum width 120 ft.” If the bridge opening is in the shape of a parabola, what is the maximum integral height that the barge can project above the water’s surface, and still fit under the bridge?

\[ \]  

6. Find the exact value of the sum of all real solutions to \(\sqrt{1 - \sqrt{1 - x}} = x\).

\[ \]