1. What is the least positive integer for which both the sum of its digits and the product of its digits are at least 10?

2. A master sushi chef, frustrated while judging a competition of apprentice chefs, angrily scrawls “LESS EEL” onto one apprentice’s unagi judging form. The master then smiles as he realizes that by adding just two symbols, the statement becomes an alphametic puzzle; i.e., $L E + S S = E E L$. Solve this puzzle, writing as your answer the 4-digit integer $L E S S$.

3. Last summer, after tripping over some exposed tree roots and skimming my knee, I got an infection from some dirt that was stuck in the wound. Fortunately, a doctor was able to clean things out and purge the infection. Solve the alphametic $\sqrt{DR} + \sqrt{R I D} = I T$, writing as your answer the 4-digit integer $D I R T$.

4. A blind woman is trying to find the bus station, located at some integer on the number line. Starting at 0, she asks a stranger which direction to walk, and the stranger says to walk in the positive direction. At each integer there is another stranger whom the blind woman asks for direction. Strangers at even integers always tell the truth, while strangers at odd integers lie the first time they are asked, and tell the truth thereafter. However, the woman assumes everyone is helpful, and follows all directions she is given. If the woman finally arrives at the bus station after traveling 101 units, at what integer is the station located?
1. What is the least positive integer for which both the sum of its digits and the product of its digits are at least 10?

No one-digit numbers work. In the teens, no numbers will satisfy the fact that the product of the digits is ≥ 10. In the 20s, we get the first acceptable product at 25, but the first sum ≥ 10 at 28.

**SOLUTIONS**

**D R I T = 2153**

Graders: All digits must be correct for credit.

2. A master sushi chef, frustrated while judging a competition of apprentice chefs, angrily scrawls “LESS EEL” onto one apprentice’s unagi judging form. The master then smiles as he realizes that by adding just two symbols, the statement becomes an alphametic puzzle; i.e., \( L E + S S = E E L \). Solve this puzzle, writing as your answer the 4-digit integer \( L E S S \).

Looking at the three-digit result (EEL), the first digit, E, must represent 1. Then the “ones” column requires that either \( 1 + S = L \), or else \( 1 + 9 = 10 \) (with \( S = 9 \) and \( L = 0 \)). In the latter case, the tens column has 1 (carried) + 9 + 9 = 11, which is false. So \( 1 + S = L \).

**Tens column:** \( L + S = 11 \) \( \rightarrow (1 + S) + S = 11 \) \( \rightarrow S = 5 \), so \( L = 6 \) and \( L E S S = 6155 \).

3. Last summer, after tripping over some exposed tree roots and skinning my knee, I got an infection from some dirt that was stuck in the wound. Fortunately, a doctor was able to clean things out and purge the infection. Solve the alphametic \( \sqrt{D R} + \sqrt[3]{R I D} = I T \), writing as your answer the 4-digit integer \( D I R T \).

Begin by looking at the three-digit perfect cubes: 125, 216, 343, 512, and 729. Throw out 343 because it contains a repeated digit. We now seek a paired units digit and hundreds digit that juxtapose to form the two-digit perfect square \( D R \). 125 \( \rightarrow 51 \) (no), 216 \( \rightarrow 62 \) (no), 512 \( \rightarrow 25 \) (yes), 729 \( \rightarrow 97 \) (no). So \( \sqrt{D R} + \sqrt[3]{R I D} = \sqrt{25} + \sqrt[3]{512} = 5 + 8 = 13 = I T \), and \( D I R T = 2153 \).

4. A blind woman is trying to find the bus station, located at some integer on the number line. Starting at 0, she asks a stranger which direction to walk, and the stranger says to walk in the positive direction. At each integer there is another stranger whom the blind woman asks for direction. Strangers at even integers always tell the truth, while strangers at odd integers lie the first time they are asked, and tell the truth thereafter. However, the woman assumes everyone is helpful, and follows all directions she is given. If the woman finally arrives at the bus station after traveling 101 units, at what integer is the station located?

Since 0 is even, the first stranger is telling the truth, and the station is located at a positive integer. The woman first walks to 1, where she is lied to, and told to walk back to 0. There, the first stranger again tells her to walk to 1, where this time she is told the truth, and walks to 2. The stranger at 2 tells her to walk to 3, where she is lied to again. This pattern can be grouped into blocks of four movements, each of the form truth/lie/truth/truth. The woman’s path is 1/0/1/2 \( \rightarrow 3/2/3/4 \) \( \rightarrow 5/4/5/6 \) \( \ldots \) etc. After every four units of movement, the woman has arrived at the next even integer. It takes 100 units to arrive at 50, so the station is located at 51.
1. If the length of the radius of a circle is increased by 100%, by what percentage will the area of the circle increase? 

$\%$  

2. In parallelogram $ABCD$ (Figure 2), point $P$ is positioned on $AD$ so that $AP = 3$ and $PD = 6$. $AC$ and $BP$ intersect at $Q$, with $PQ = 2$. Determine exactly the length $BQ$. 

$BQ =$  

3. The triangle in Figure 3 shows two nested right triangles that share acute angle $\theta$: the smaller with leg of length 1 and hypotenuse of length $x$; the larger with leg lengths $h$ and $2x$. If $\sin \theta = \frac{1}{3}$, determine $h$ exactly. 

$h =$  

4. In $\triangle JKL$ (Figure 4), points $X, Y,$ and $Z$ divide sides $LK$, $JL$, and $JF$ respectively, each in the ratio 1:2. Also, intersection points $N_1$, $N_2$, and $N_3$ divide each of the cevians $JX$, $KY$, and $JZ$ in the ratio 3:3:1; i.e., $JN_1:N_1N_2:N_2X = 3:3:1$, and similarly for $KY$ and $JZ$. If the area of $\triangle JKL$ is $m$, express the area of $\triangle N_1N_2N_3$ in terms of $m$. 

$\text{Area}\left[\triangle N_1N_2N_3\right] =$  

Name: ____________________________  
Team: ____________________________
1. If the length of the radius of a circle is increased by 100%, by what percentage will the area of the circle increase?

The length of the radius is doubling, so the area will become four times as big. If the original area was $A$, then the new area is $4A$; this is an increase of $3A$, or 300%.

2. In parallelogram $ABCD$ (Figure 2), point $P$ is positioned on $AD$ so that $AP = 3$ and $PD = 6$. $AC$ and $BP$ intersect at $Q$, with $PQ = 2$. Determine exactly the length $BQ$.

$$\triangle APQ \sim \triangle CBQ \text{ by AA, so } \frac{PQ}{QB} = \frac{AP}{BC} \implies \frac{2}{QB} = \frac{3}{9} \implies QB = 6.$$ 

3. The triangle in Figure 3 shows two nested right triangles that share acute angle $\theta$: the smaller with leg of length 1 and hypotenuse of length $x$; the larger with leg lengths $h$ and $2x$. If $\sin \theta = \frac{1}{3}$, determine $h$ exactly.

Because $\sin \theta = \text{opposite/hypotenuse}$, we can conclude that $x = 3$. Then, use the Pythagorean Theorem to calculate that the third side length of the smaller triangle is $2\sqrt{2}$. The two triangles are similar by AA, so set up a proportion: $$\frac{2\sqrt{2}}{1} = \frac{6}{h} \implies h = \frac{6}{2\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}.$$ 

4. In $\triangle JKL$ (Figure 4), points $X$, $Y$, and $Z$ divide sides $\overline{JK}$, $\overline{JL}$, and $\overline{JY}$ respectively, each in the ratio 1:2. Also, intersection points $N_1$, $N_2$, and $N_3$ divide each of the cevians $\overline{JN_1}$, $\overline{KN_1}$, and $\overline{LN_1}$ in the ratio 3:3:1; i.e., $JN_1 : N_1N_2 : N_2X = 3:3:1$, and similarly for $\overline{KN_1}$ and $\overline{LN_1}$. If the area of $\triangle JKL$ is $m$, express the area of $\triangle N_1N_2N_3$ in terms of $m$.

Drop altitudes from $N_2$ and $J$ as shown. $\triangle JXL$ has the same height as $\triangle JKL$ but one-third the base, so its area is $m/3$. (Same for $\triangle LZK$ and $\triangle KY$.) $\triangle N_2XL$ has the same base as $\triangle JXL$ but one-seventh the height, so its area is $m/21$. The area we want can be calculated by taking $\triangle JKL$, subtracting out $\triangle JXL$, $\triangle LZK$, and $\triangle KY$, then adding back in $\triangle N_2XL$, $\triangle N_3ZK$, and $\triangle N_3YJ$, each of which was subtracted twice: $m - 3(m/3) + 3(m/21) = m/7$. 

1. 

$$BQ = 6$$ 

2. 

$$h = \frac{3\sqrt{2}}{2}$$ 

3. 

$$\text{Area}\left[\triangle N_1N_2N_3\right] = \frac{1}{7}m$$ or equivalent 

4. 

Drop altitudes from $N_2$ and $J$ as shown. $\triangle JXL$ has the same height as $\triangle JKL$ but one-third the base, so its area is $m/3$. (Same for $\triangle LZK$ and $\triangle KY$.) $\triangle N_2XL$ has the same base as $\triangle JXL$ but one-seventh the height, so its area is $m/21$. The area we want can be calculated by taking $\triangle JKL$, subtracting out $\triangle JXL$, $\triangle LZK$, and $\triangle KY$, then adding back in $\triangle N_2XL$, $\triangle N_3ZK$, and $\triangle N_3YJ$, each of which was subtracted twice: $m - 3(m/3) + 3(m/21) = m/7$. 

Minnesota State High School Mathematics League 
2013-14 Meet 5, Individual Event B

SOLUTIONS
1. How many positive odd integers are less than 300?

2. Given that $R = \frac{501! - 499!}{500!}$, $S = \frac{500! - 498!}{500!}$, and $T = \frac{500! - 498!}{499!}$, place the variables $R$, $S$, and $T$ into the proper boxes to make the inequality true.

3. How many positive odd integers are less than 300 and have all digits distinct?

4. Three married couples on a cruise together are perusing a pamphlet that describes three different shore excursions. Collectively, they want to "try everything", so at least one person goes on each excursion. In how many ways can this be done if no one wants to be on the same excursion with his/her spouse?
1. How many positive odd integers are less than 300?

The odd integers \( \{1, 3, 5, \ldots, 299\} \) are all one less than \( \{2, 4, 6, \ldots, 300\} \), which are all double \( \{1, 2, 3, \ldots, 150\} \). All three sets have the same number of elements, which is 150.

2. Given that \( R = \frac{501! - 499!}{500!} \), \( S = \frac{500! - 498!}{500!} \), and \( T = \frac{500! - 498!}{499!} \), place the variables \( R, S, \) and \( T \) into the proper boxes to make the inequality true.

\[
R = \frac{501! - 499!}{500!} = \frac{501!}{500!} \frac{499!}{500!} - \frac{1}{500}; \\
S = \frac{500! - 498!}{500!} = \frac{500!}{500!} \frac{498!}{500!} - \frac{1}{499\cdot500}; \\
T = \frac{500! - 498!}{499!} = \frac{500!}{499!} \frac{498!}{499!} - \frac{1}{499}.
\]

\( S \) is slightly less than 1, \( T \) slightly less than 500, etc.

3. How many positive odd integers are less than 300 and have all digits distinct?

There are 5 one-digit odd integers. The two-digit odd integers can be built either by placing one of four non-zero even digits in the tens place and an odd digit in the units place \( (4 \cdot 5 = 20 \text{ ways}) \), or by placing an odd digit in the tens place and one of the four remaining odd digits in the units place \( (5 \cdot 4 = 20 \text{ ways}) \), for a total of 40 two-digit possibilities. The three-digit odd integers will either consist of a 1 in the hundreds place, any of four remaining odd digits in the units place, and any of the eight remaining digits in the tens place \( (1 \cdot 8 \cdot 4 = 32 \text{ ways}) \), or a 2 in the hundreds place, any odd digit in the units place, and one of eight remaining digits in the tens place \( (1 \cdot 8 \cdot 5 = 40 \text{ ways}) \). This makes 72 three-digit odds, and \( 5 + 40 + 72 = 117 \) overall.

4. Three married couples on a cruise together are perusing a pamphlet that describes three different shore excursions. Collectively, they want to “try everything”, so at least one person goes on each excursion. In how many ways can this be done if no one wants to be on the same excursion with his/her spouse?

The maximum of people per excursion is 3 (one from each couple). This means the distribution of people among excursions must either be 3/2/1 or 2/2/2. In the 3/2/1 case, we first place the 3: choose one person from each couple \( (2^3) \), and which excursion they go on \( (3 \text{ choices}) \). Then, select two of the remaining three people \( (\binom{3}{2} = 3 \text{ ways}) \), and which remaining excursion \( (2 \text{ choices}) \) they choose. The last person is placed by default, for a total of \( 8 \cdot 3 \cdot 3 \cdot 2 = 144 \text{ ways} \). In the 2/2/2 case, first pair one member of couple A with a member of couples B or C \( (4 \text{ ways}) \); then, the other \( A \) must pair with someone from the yet-unpaired couple \( (2 \text{ ways}) \). The last pair is forced. Now place the pairs on excursions \( (\binom{3}{2} = 6) \), for a total of \( 4 \cdot 2 \cdot 6 = 48 \text{ ways} \). Altogether, we have \( 144 + 48 = 192 \).
Mr. Blue measures everything using units of his own height, which he calls “one blue”. He has a backyard pool, shaped like a rectangular prism, which is 5 blues wide, 10 blues long, and 2 blues deep. If one blue = 2 meters, what is the volume of the pool in cubic meters?

\[ V = \text{__________} \text{ m}^3 \]

Jo begins writing the positive integers as one long string of digits:

1234567891011121314 …

What will be the 500th such digit she writes?

\[ x = \text{__________} \]

Integers \( x \) and \( y \) satisfy the equation \( x^2 + y^2 + 12 = 4x - 6y \).

List all possible values for \( x \).

\[ x = \text{__________} \]

If \( a \), \( b \), and \( c \) are positive real numbers such that \( a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 4 \), determine exactly the maximum possible value for \( c \).

\[ c_{\text{max}} = \text{__________} \]

NO CALCULATORS are allowed on this event.

Name: ____________________________

Team: ____________________________
NO CALCULATORS are allowed on this event.

\[ V = 800 \text{ m}^3 \]

1. Mr. Blue measures everything using units of his own height, which he calls “one blue”. He has a backyard pool, shaped like a rectangular prism, which is 5 blues wide, 10 blues long, and 2 blues deep. If one blue = 2 meters, what is the volume of the pool in cubic meters?

The volume is \( (5)(10)(2) = 100 \) cubic blues, or \( 100(2^3) = 800 \) cubic meters.  

[2013 AMC 12B, problem #2]

2. Jo begins writing the positive integers as one long string of digits:

\[ 1234567891011121314 \ldots \]

What will be the 500th such digit she writes?

It takes Jo 9 digits to get through the one-digit numbers. Then 10-19, 20-29, etc. each require 20 digits, or a total of 180 digits to get through the two-digit numbers. There are 100 numbers from 100-199, each requiring 3 digits; that’s another 300 digits. She’s at \( 9 + 180 + 300 = 489 \) so far. Continue writing: 200, 201, 202, 203, \ldots  

The 500th digit is 0.  

[2013 AMC 12B, problem #7]

\[ x = 1, 2, 3 \]

3. Integers \( x \) and \( y \) satisfy the equation \( x^2 + y^2 + 12 = 4x - 6y \).

List all possible values for \( x \).

Move all terms to the left side and complete the square:

\[
(x^2 - 4x) + (y^2 + 6y) + 12 = 0 \Rightarrow (x - 2)^2 - 4 + (y + 3)^2 - 9 + 12 = 0 \Rightarrow (x - 2)^2 + (y + 3)^2 = 1, \text{ which describes a circle of radius 1 centered at } (2, -3). \]

The \( x \)-coordinates of this circle will be within 1 unit of 2 (1 \( \leq x \leq 3 \)), so the only possible integral values are \( x = 1, 2, \) and 3.

[2013 AMC 12B, problem #6]

\[ c_{\text{max}} = 2 + \sqrt{3} \]

4. If \( a, b, \) and \( c \) are positive real numbers such that \( a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 4 \), determine exactly the maximum possible value for \( c \).

Consider \( a + \frac{1}{a} \). This quantity must always be at least 2. (There are numerous simple proofs of this; some excellent pictorial ones: http://www.maa.org/sites/default/files/Nelsen99959975.pdf)

Argue similarly for \( b + \frac{1}{b} \). Then \( \left( a + \frac{1}{a} \right) + \left( b + \frac{1}{b} \right) \geq 4 \), \( \left( a + \frac{1}{a} \right) + \left( b + \frac{1}{b} \right) + \left( c + \frac{1}{c} \right) = 8 \), so it must be true that \( \left( c + \frac{1}{c} \right) \leq 4 \). Solving the equality portion: \( c^2 + 1 = 4c \Rightarrow c^2 - 4c + 1 = 0 \Rightarrow c = \frac{4 \pm \sqrt{16 - 4}}{2} = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}, \text{ the greater value of which is } 2 + \sqrt{3} \).

[2013 AMC 12B, problem #17]
Each question is worth 4 points. Team members may cooperate in any way, but at the end of 20 minutes, submit only one set of answers. Place your answer to each question on the line provided.

**Note:** For alphametics problem #1 below, each letter represents a different digit, and leading digits may not be zero.

1. The authorities are investigating an electronic bank heist where the culprits dumped the stolen money into a private account. So far, they've determined that a text message sent from one robber to another ("DEED DONE") is actually an alphametic puzzle, with 

\[ (DE)(ED) = DONE, \]

and \( D O N E \) the key PIN code needed to access the account. If it's already known that \( N = 5 \), write the 4-digit PIN code \( D O N E \).

2. A square is inscribed in a circle. Then, the circle is divided into two congruent semicircles by a diameter, and a smaller square is inscribed in one of the semicircles. Determine exactly the ratio of the squares' areas (smaller : larger).

3. What is the least positive integer whose digits have a product of at least 1000?

4. A certain lock has five numbered buttons and is unlocked by an ordered sequence of three “moves”. Each “move” consists of simultaneously depressing 1, 2, or 3 of the buttons. Once a button is depressed during any move of the unlocking sequence, it remains depressed and may not be used again in further moves. Keeping in mind that not all five buttons need be used during an unlocking sequence, how many sequences are possible?

5. Ollie the Occasional Liar never lies twice in a row. When asked for his 5-digit ZIP code (ranging between 00000 and 99999), he made the following statements, in order:
   - “The sum of the digits is at most 25.”
   - “The product of the digits is not a perfect square.”
   - “The sum of the digits is at most 13.”
   - “The product of the digits is a perfect square.”
   - “The sum of the digits is at most 19.”
   - “The product of the digits is not a perfect square.”
   - “No digit appears more than twice.”
If Ollie’s ZIP code is treated as a 5-digit integer, what is its greatest possible value?

6. Blair and his father computed the arithmetic and geometric means of their ages. The results were 36 and 39, in some order. How old is Blair?

Blair =
1. The authorities are investigating an electronic bank heist where the culprits dumped the stolen money into a private account. So far, they’ve determined that a text message sent from one robber to another (“DEED DONE”) is actually an alphametic puzzle, with \((DE)(ED) = DONЕ\), and \(DONЕ\) the key PIN code needed to access the account. If it’s already known that \(N = 5\), write the 4-digit PIN code \(DONЕ\).

**Graders:** Deduct 1 point for the answer 5:2.

2. A square is inscribed in a circle. Then, the circle is divided into two congruent semicircles by a diameter, and a smaller square is inscribed in one of the semicircles. Determine exactly the ratio of the squares’ areas (smaller : larger).

3. What is the least positive integer whose digits have a product of at least 1000?

4. A certain lock has five numbered buttons and is unlocked by an ordered sequence of three “moves”. Each “move” consists of simultaneously depressing 1, 2, or 3 of the buttons. Once a button is depressed during any move of the unlocking sequence, it remains depressed and may not be used again in further moves. Keeping in mind that not all five buttons need be used during an unlocking sequence, how many sequences are possible?

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   - “The sum of the digits is at most 25.”
   - “The product of the digits is not a perfect square.”
   - “The sum of the digits is at most 13.”
   - “The product of the digits is a perfect square.”
   - “The sum of the digits is at most 19.”
   - “The product of the digits is not a perfect square.”
   - “No digit appears more than twice.”

If Ollie’s ZIP code is treated as a 5-digit integer, what is its greatest possible value?

6. Blair and his father computed the arithmetic and geometric means of their ages. The results were 36 and 39, in some order. How old is Blair?
1. The units digits (E \times D \times E) indicate that either E = 0, D = 1, E is odd and D = 5, or D is even and E = 6. E is the tens digit of the second factor in the product, so it cannot be 0. If D = 5, our only chance to end up with a product \geq 5000 is 59 \times 95 = 5605, which does not have the required 5 in the tens place. If E = 6, our only chance is 86 \times 68 = 5848, which fails again. So D = 1, and (10 + E) \times 10 = 1000 \times 100 + 50 + E \Rightarrow 10E^2 + 101E + 10 = 1000 \times 10 - 1040. To get a tens digit of 4 on the left side, E = 2 or E = 8; E = 2 makes the final product less than 1000, so E = 8, and 18 \times 81 = 1458.

2. See Figure 2. In the semicircle, \( s^2 + \left( \frac{5}{2} \right)^2 = r^2 \Rightarrow \frac{5}{4} s^2 = r^2 \Rightarrow \). In the full circle, \( x^2 + x^2 = r^2 \Rightarrow 2x^2 = r^2 \Rightarrow 4x^2 = \left( 2x \right)^2 = \text{Area} = 2r^2 \). The ratio of areas is \( \frac{4}{5}r^2 : 2r^2 = 2 : 5 \).

3. 1000 = 10^3, so no 3-digit integer will work. Also, a valid 4-digit integer cannot contain any 1’s, as those will have no impact on the product, and render the integer equivalent (for our purposes) to a 3-digit integer. So: start with an integer of the form \( 2 A B C \). We need \( A \cdot B \cdot C \geq 500 \), with A as small as possible. \( A = 5 \Rightarrow B \cdot C \geq 100 \); \( A = 6 \Rightarrow B \cdot C \geq 83.33 \ldots \), but the largest \( B \cdot C \) can be is 81. Thus \( A = 7 \Rightarrow B \cdot C \geq 71.42 \ldots \), and we choose \( B = 8, C = 9 \) to form the integer 2789.

4. Make a table where each column represents a set of related sequences. For each (C)ombination entry in the table, the first number is the # of remaining undepressed buttons, and the second number is the # of buttons depressed during that move:

<table>
<thead>
<tr>
<th>Move #1</th>
<th>C_3</th>
<th>C_2</th>
<th>C_2</th>
<th>C_2</th>
<th>C_1</th>
<th>C_1</th>
<th>C_1</th>
<th>C_1</th>
<th>C_1</th>
<th>C_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Move #2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Move #3</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td># of ways</td>
<td>20</td>
<td>30</td>
<td>30</td>
<td>60</td>
<td>20</td>
<td>30</td>
<td>60</td>
<td>20</td>
<td>60</td>
<td>60</td>
</tr>
</tbody>
</table>

The sum of the entries in the last row = 390.

5. Focus first on statements #2, #4, and #6, which contain contradictions. If #2 is true, then #4 is false and #6 is true. Since #4 is false and Ollie will not lie twice in a row, both #3 and #5 must be true. #3 is more limiting than both #1 and #5, so we seek a ZIP code, the product of whose digits is not a perfect square, and the sum of whose digits is at most 13. At this point, we might start with 91111, but the product of digits is a perfect square. Same with 82111. (Note that we can never use 0 as a digit here.) 81111 is a possibility, as long as #7 is false. If #7 is true (no digit appears more than twice), then we’re further restricted; 72211 is the best we can do.

What if #2 had been false? Then #4 is true and #6 is false, and all other statements must be true, as they are adjacent to either #2 or #6. So we now seek a ZIP code, the product of whose digits is a perfect square, the sum of whose digits is at most 13, where no digit appears more than twice. We have a winner: 93100.

6. The famous AM-GM inequality (look it up!) states that the geometric mean of two numbers can never exceed the arithmetic mean. Therefore, the arithmetic mean must be 39, and the geometric mean 36. Set up a system of equations, and solve:

\[
\begin{align*}
\frac{B + F}{2} &= 39 \\
\sqrt{BF} &= 36
\end{align*}
\Rightarrow \begin{align*}
B + F &= 78 \\
BF &= 1296
\end{align*} \Rightarrow B(78 - B) = 1296 \Rightarrow B^2 - 78B + 1296 = (B - 54)(B - 24) = 0 \Rightarrow B = 24, F = 54.