Minnesota State High School Mathematics League
2014-15 State Tournament, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this Tournament event.

**NO CALCULATORS are allowed on this event.**

1. Determine exactly the value of \( \frac{3^4 \cdot 4^2}{2^6 \cdot 6^2} \).

2. Solve for the digits A, B, C, and D if \( A A A A A - B B B B + C C + D = 1234 \).
   
   \( A = \) ________
   
   \( B = \) ________
   
   \( C = \) ________
   
   \( D = \) ________

3. If \( x + y \leq 10 \), \( y + z \leq 14 \), and \( x + z \geq 12 \), determine exactly the **minimum** value of \( z - y \).
   
   \( \text{min}(z - y) = \) ________

4. Find all pairs of positive integers \( (x, y) \) for which \( x^2 - 144 = 25y^2 \).
   
   \( (x, y) = \) ________

Name: _______________________________  Team: _______________________________
1. Determine exactly the value of \( \frac{3^3 \cdot 4^4}{2^6 \cdot 6^2} \).

\[
\frac{3^3 \cdot 4^4}{2^6 \cdot 6^2} = \frac{3^3 \cdot (2^2)^4}{2^6 \cdot (2 \cdot 3)^2} = \frac{3^3 \cdot 2^8}{2^6 \cdot 2^2 \cdot 3^2} = \frac{3^3 \cdot 2^8}{2^6 \cdot 3^2} = 3.
\]

2. Solve for the digits \( A, B, C, \) and \( D \) if \( A A A A - B B B + C C + D = 1234 \).

\[
\begin{align*}
A & = 2 \\
B & = 9 \\
C & = 1 \\
D & = 0
\end{align*}
\]

Graders:
All values must be correct.

3. If \( x + y \leq 10, \ y + z \leq 14, \) and \( x + z \geq 12, \) determine exactly the minimum value of \( z - y \).

\[
\text{Isolate } x \text{ in the first and third inequalities to obtain } x \leq 10 - y \text{ and } x \geq 12 - z \text{ respectively. This means that } x \text{ is at least } 12 - z, \text{ but no more than } 10 - y. \text{ This suggests the inequality } 12 - z \leq 10 - y \Rightarrow 2 \leq z - y. \text{ Therefore, the minimum value of } z - y \text{ is 2.}
\]

4. Find all pairs of positive integers \( (x, y) \) for which \( x^2 - 144 = 25y^2 \).

\[
\text{Rearranging and factoring the given equation yields } x^2 - 25y^2 = (x - 5y)(x + 5y) = 144. \text{ We are looking for two positive integer factors whose product is 144. Notice that these factors are 10y apart. The factor pairs that are a multiple of ten apart are (2, 72), (8, 18), and (12, 12). However, } y \neq 0, \text{ excluding (12, 12). The other two pairs result in the systems }
\]
\[
\begin{align*}
\begin{cases}
\text{ } x + 5y = 72 \\
x - 5y = 2
\end{cases}
\text{ and } \begin{cases}
\text{ } x + 5y = 18 \\
x - 5y = 8
\end{cases}
\end{align*}
\]
_solving each of these yields \( (x,y) = (37,7), (13,1). \)
1. In Figure 1, $\triangle ABC$ and $\triangle ADE$ are both isosceles, each with an apex angle at $A$. If $\angle BAC = 82^\circ$ and $\angle ADE = 71^\circ$, what is the measure of $\angle DAB$?

2. One angle of a rhombus measures $144^\circ$, and its shorter diagonal is $5$ cm long. Calculate the length of the other diagonal.

3. Three circles of radius $2$ are drawn as shown in Figure 3, each with its center lying on an intersection point of the other two circles. $P$, $Q$, and $R$ are the centers of the three circles. Determine exactly the area of the shaded interior region.

4. $WXYZ$ is a square of side length $1$ (Figure 4). Point $K$ lies between $Y$ and $Z$ on $YZ$, and points $L$ and $M$ lie on $WK$ such that $XL \perp WK$ and $\triangle WZK \sim \triangle XLW \sim \triangle XLM$. If the area of $\triangle XLM$ is exactly $n$ times the area of $\triangle WZK$, express using an inequality all possible values for $n$.

Place your answer to each question on the line provided. You have 15 minutes for this Tournament event.

Name: ___________________________ 
Team: ___________________________
1. In Figure 1, \( \triangle ABC \) and \( \triangle ADE \) are both isosceles, each with an apex angle at \( A \). If \( m \angle BAC = 82^\circ \) and \( m \angle ADE = 71^\circ \), what is the measure of \( \angle DAB \)?

\[
m \angle DAE = 180^\circ - 2 \cdot 71^\circ = 38^\circ, \text{ so } m \angle DAB = \frac{1}{2}(82^\circ - 38^\circ) = 22^\circ.
\]

2. One angle of a rhombus measures 144°, and its shorter diagonal is 5 cm long. Calculate the length of the other diagonal.

Remembering that the diagonals of a rhombus meet at a right angle, and that its interior angles are supplementary, we have:

\[
\tan 18^\circ = \frac{2.5}{d} \Rightarrow d = \frac{2.5}{\tan 18^\circ} \Rightarrow 2d = \frac{5}{\tan 18^\circ} = 15.388 \text{ cm}.
\]

3. Three circles of radius 2 are drawn as shown in Figure 3, each with its center lying on an intersection point of the other two circles. \( P \), \( Q \), and \( R \) are the centers of the three circles. Determine exactly the area of the shaded interior region.

Form equilateral \( \triangle PQR \) with side length 2. Then the area of the shaded region is the area of \( \triangle PQR \) plus the area of three 60° segments, or equivalently, the area of three 60° sectors minus twice the area of \( \triangle PQR \):

\[
3 \left( \frac{1}{2} \cdot 2 \cdot 2 \right) - 2 \left( \frac{2^2 \cdot \sqrt{3}}{4} \right) = 2\pi - 2\sqrt{3}.
\]

4. \( WXYZ \) is a square of side length 1 (Figure 4). Point \( K \) lies between \( Y \) and \( Z \) on \( YZ \), and points \( L \) and \( M \) lie on \( WK \) such that \( XL \perp WK \) and \( \triangle WZK \sim \triangle XLM \). If the area of \( \triangle XLM \) is exactly \( n \) times the area of \( \triangle WZK \), express using an inequality all possible values for \( n \). \([2014\ AMC\ 12B, \#21]\)

Let \( ZK = x \). Then if we set \( XL = a \), \( WL = LM = ax \). From the given area relationship, we know that \( \frac{1}{2} \cdot (ax) \cdot a = n \cdot \left( \frac{1}{2} \cdot x \cdot 1 \right) \Rightarrow n = a^2 \). Because \( XLM \) is a right triangle, \( a^2 + (ax)^2 = 1 \Rightarrow a^2 = \frac{1}{1 + x^2} \), and so \( n = \frac{1}{1 + x^2} \Rightarrow x = \sqrt{\frac{1}{n} - 1} \).  

\( 0 < x < 1 \), so \( 0 < \frac{1}{n} - 1 < 1 \Rightarrow 1 < \frac{1}{n} < 2 \Rightarrow \frac{1}{2} < n < 1 \).
Minnesota State High School Mathematics League
2014-15 State Tournament, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this Tournament event.

**NO CALCULATORS are allowed on this event.**

1. Determine exactly the infinite sum $6 + \left( -4 \right) \frac{8}{3} + \left( -\frac{16}{9} \right) + \ldots$

2. In a certain country, 10% of the population have blue eyes, and 30% of the population have blond hair. It is also known that 20% of blonds have blue eyes. What percentage of people with blue eyes are blond?

3. For what acute angle (in degrees) does $\cos 68^\circ \cos 60^\circ = \cos 11^\circ \cos A$?

4. The product $7(777 \ldots 7)$, where the second factor has $k$ digits, results in an integer whose digits sum to 997. What is the value of $k$?

Name: ____________________________  Team: ____________________________
NO CALCULATORS are allowed on this event.

1. Determine exactly the infinite sum: $6 + \left(-\frac{4}{3}\right) + \left(-\frac{16}{9}\right) + \ldots$

The explicit formula for the sum of an infinite geometric series is $S = \frac{a}{1-r}$.

Here, the common ratio is $\frac{-4}{6} = -\frac{2}{3}$; therefore, $S = \frac{6}{1-\left(-\frac{2}{3}\right)} = \frac{6\cdot\frac{3}{5}}{\frac{5}{3}} = \frac{18}{5} = \frac{3\cdot3}{5}$.

2. In a certain country, 10% of the population have blue eyes, and 30% of the population have blond hair. It is also known that 20% of blonds have blue eyes. What percentage of people with blue eyes are blond?

If 20% of the blonds have blue eyes, then $0.20(0.30) = 0.06 = 6\%$ of the population are blond with blue eyes. Therefore, $\frac{6\%}{10\%} = 0.6$, or 60\%, of the people with blue eyes are blond.

$m\angle A = \boxed{79^\circ}$

3. For what acute angle (in degrees) does $\cos 68^\circ \cos 60^\circ = \cos 11^\circ \cos A$?

$\cos 60^\circ = \frac{1}{2}$, so $\frac{1}{2} \cos 68^\circ = \cos 11^\circ \cos A \Rightarrow \cos 68^\circ = 2 \cos 11^\circ \cos A$. Using a cofunction identity on both $\cos 68^\circ$ and $\cos A$, we obtain $\sin 22^\circ = 2 \cdot \cos 11^\circ \sin (90^\circ - A)$. This fits the format of the double-angle formula for sine: $\sin 22^\circ = 2 \cdot \cos 11^\circ \sin 11^\circ \Rightarrow A = 79^\circ$.

$k = \boxed{247}$

4. The product $7(777 \ldots 7)$, where the second factor has $k$ digits, results in an integer whose digits sum to 997. What is the value of $k$? 

Start by seeing if a pattern emerges. If the second factor only had one seven, that would give $7(7) = 49$ and the sum of digits would be $4 + 9 = 13$. If the second factor is 77, then the product is 539 and the digit sum is 17. Continuing, we see that $7(777) = 5439$ with a digit sum of 21, and $7(7777) = 54439$ with a digit sum of 25. Each additional “7” digit in the second factor contributes 4 more to the sum. The sequence of digit sums is arithmetic: 13, 17, 21, 24, …, so we can set up the following equation: $997 = 13 + 4(k-1) \Rightarrow 984 = 4(k-1) \Rightarrow k = 247$. 

[2014 AMC 12A, #16]
1. Determine exactly the greater of the two roots of $x^2 - 6x + 9 = 2$.

\[ b = \text{________________} \] 2. Determine exactly the base $b > 0$ for which $\log_2(\log_b 3) = 4$.

\[ f(-3) + f(3) = \text{________________} \] 4. Let $f$ be a cubic polynomial where $f(-1) = -k$, $f(0) = k$, and $f(1) = 5k$. Express $f(-3) + f(3)$ in terms of $k$. 

Name: _______________________________ Team: _______________________________
1. Determine exactly the greater of the two roots of \( x^2 - 6x + 9 = 2 \).

\[
x^2 - 6x + 7 = 0 \implies x = \frac{6 \pm \sqrt{36 - 28}}{2} = \frac{6 \pm 2\sqrt{2}}{2} = 3 \pm \sqrt{2}, \text{ the greater of which is } 3 + \sqrt{2}.
\]

2. Determine exactly the base \( b > 0 \) for which \( \log_2 (\log_b 3) = 4 \).

\[
\log_2 (\log_b 3) = 4 \implies \log_b 3 = 2^4 \implies b^{16} = 3 \implies b = 16^{\frac{1}{3}}.
\]

3. An ellipse with foci located at \((0, 1)\) and \((4, 3)\) passes through the origin. What is the length of the ellipse's minor axis?

This is an oblique ellipse (oriented on a diagonal), but we need not get discouraged. The locus definition of an ellipse requires that the sum of the distances from any point on the ellipse to the two foci be a constant. Calculating the distances from \((0, 0)\) to \((0, 1)\) and \((4, 3)\), we obtain a common sum of \(1 + \sqrt{4^2 + 3^2} = 1 + 5 = 6\), which is also the length of the major axis (\(= 2a\)). The ellipse is centered at \((2, 2)\), so we have a focal length of \(c = \sqrt{(4-2)^2 + (3-2)^2} = \sqrt{2^2 + 1^2} = \sqrt{5}\), and by the Pythagorean relationship, \(c^2 = a^2 - b^2 \implies b = \sqrt{a^2 - c^2} = \sqrt{3^2 - (\sqrt{5})^2} = \sqrt{9 - 5} = 2 \implies 2b = 4\).

4. Let \( f \) be a cubic polynomial where \( f(-1) = -k, f(0) = k, \) and \( f(1) = 5k \). Express \( f(-3) + f(3) \) in terms of \( k \).

\[
\begin{align*}
f(-3) + f(3) &= 20k \\
\text{Let } f(x) &= ax^3 + bx^2 + cx + d. \text{ Then } f(-1) &= a(-1)^3 + b(-1)^2 + c(-1) + d = -a + b - c + d = -k, \\
f(0) &= a(0)^3 + b(0)^2 + c(0) + d = d = k, \text{ and } f(1) &= a(1)^3 + b(1)^2 + c(1) + d = a + b + c + d = 5k. \\
\text{Adding the first and third equations together yields } 2b + 2d = 4k, \text{ and since } d = k, b = k \text{ also.}
\end{align*}
\]

Now consider: \( f(-3) + f(3) = (-27a + 9b - 3c + d) + (27a + 9b + 3c + d) = 18b + 2d = 20k \).
Each question is worth 4 points. Team members may cooperate in any way, but at the end of 30 minutes, submit only one set of answers. Place your answer to each question on the line provided.

1. Figure 1 shows right triangles $ABC$, $ABF$, $BCF$, $ADE$, and $CEG$ with $m\angle G = m\angle DAE = 30^\circ$. If $AB = 13$ and $AD = DG$, calculate the length $EF$.

2. Three positive integers have a least common multiple of 360 and a greatest common divisor of 2. What are the smallest and largest possible values for the product of the three integers?

3. If $\alpha$ and $\beta$ are acute angles for which $\sin(\alpha + \beta) = 5 \sin \alpha \cos \beta = \cos (\alpha - \beta)$, determine exactly all possible values of $\tan (\alpha + \beta)$.

4. The area of a triangle whose altitudes have lengths 3, 4, and 5 can be expressed in the form $\frac{k}{\sqrt{a\cdot b\cdot c\cdot d}}$, where $k$ is a perfect square and $a, b, c, d$ are prime integers. Do so.

5. Write the equations of all lines that pass through the point $(2, 3)$ and which are a distance of 2 from the origin.

6. The graph of the function $f(x)$ is a parabola that passes through $(-3, 14)$ and whose minimum value is achieved at $x = 20$. What is the smallest positive integer $c$ such that the minimum value of $f(f(x))$ could potentially be achieved at $x = c$?

Team: 

Minnesota State High School Mathematics League
2014-15 State Tournament, Team Event
1. Figure 1 shows right triangles ABC, ABF, BCF, ADE, and CEG with $m\angle G = m\angle DAE = 30^\circ$. If $AB = 13$ and $AD = DG$, calculate the length $EF$.

2. Three positive integers have a least common multiple of 360 and a greatest common divisor of 2. What are the smallest and largest possible values for the product of the three integers?

   smallest = \[
   \frac{1440}{2}
   \]

   largest = \[
   \frac{259200}{2}
   \]

   Graders:
   Award 2 points for each correct value.

3. If $\alpha$ and $\beta$ are acute angles for which $\sin(\alpha + \beta) = 5 \sin \alpha \cos \beta = \cos(\alpha - \beta)$, determine exactly all possible values of $\tan(\alpha + \beta)$.

   $\tan(\alpha + \beta) = \pm \frac{5}{3}$

   or $\pm1 \frac{2}{3}$ or $\pm1.6$

   or equivalent $\pm\frac{3}{1}$

   Graders:
   Award 2 points for each correct value.

4. The area of a triangle whose altitudes have lengths 3, 4, and 5 can be expressed in the form $\frac{k}{\sqrt{a \cdot b \cdot c \cdot d}}$, where $k$ is a perfect square and $a, b, c, d$ are prime integers. Do so.

   Area = \[
   \frac{3600}{\sqrt{7 \cdot 17 \cdot 23 \cdot 47}}
   \]

5. Write the equations of all lines that pass through the point (2, 3) and which are a distance of 2 from the origin.

   $x = 2$

   or equivalent $y = \frac{5}{12}x + \frac{13}{6}$

   Graders:
   Award 2 points for each correct value.

6. The graph of the function $f(x)$ is a parabola that passes through $(-3, 14)$ and whose minimum value is achieved at $x = 20$. What is the smallest positive integer $c$ such that the minimum value of $f(f(x))$ could potentially be achieved at $x = c$?

   $c = 44$
1. Set \( AD = DG = x \). Then \( DE = \frac{x}{2} \) and \( AE = \frac{x\sqrt{3}}{2} \). \( EG = \frac{3x}{2} \), so \( CE = \frac{x}{3} \) \( \cdot \) \( EG = \frac{3x\sqrt{3}}{2} \) also. Because \( AB = 13 \), \( AC = 26 = x\sqrt{3} \), and \( x = \frac{26\sqrt{3}}{3} \). Finally, \( AF = \frac{13}{2} \), so \( EF = AC - AF - CE = 26 - \frac{13}{2} - \frac{3\sqrt{3}}{2} = 26 - \frac{13}{2} - 13 = \frac{13}{2} \).

2. Observe that \( 360 = 2^3 \cdot 3^2 \cdot 5^1 \). At least one of the integers must have 8 as a factor, while the other two will each have only a single factor of 2. Also, at least one of the integers must have factors of 9 and 5. This results in a product of \( 1440 \), achieved for the integers 2, 3, and 360.

To generate the largest possible product, give two integers a factor of 8, with the third integer having only a single factor of 2. Similarly, at least one integer must not be divisible by 3, with the other two divisible by 9, and at least one integer is not divisible by 5, but the other two both can be divisible by 5. This results in the product \( 259200 \), achieved for 2, 360, 360.

3. \( \sin(\alpha + \beta) = \cos(\alpha - \beta) \Rightarrow (\alpha + \beta) + (\alpha - \beta) = 2\alpha = 90^\circ \), so \( \alpha = 45^\circ \), or \( (\alpha + \beta) - (\alpha - \beta) = 2\beta = 90^\circ \), so \( \beta = 45^\circ \).

\[ \sin(45^\circ + \beta) = 5 \sin 45^\circ \cos \beta \Rightarrow \frac{\sqrt{2}}{2} \cos \beta + \frac{\sqrt{2}}{2} \sin \beta = 5 \cdot \frac{\sqrt{2}}{2} \cos \beta \Rightarrow \cos \beta + \sin \beta = 5 \cos \beta \Rightarrow \tan \beta = 4, \tan \alpha = 1; \]
\[ \sin(\alpha + 45^\circ) = 5 \sin \alpha \cos 45^\circ \Rightarrow \sin \alpha \cdot \frac{\sqrt{2}}{2} + \cos \alpha \cdot \frac{\sqrt{2}}{2} = 5 \sin \alpha \cdot \frac{\sqrt{2}}{2} \Rightarrow \sin \alpha + \cos \alpha = 5 \sin \alpha \Rightarrow \tan \alpha = \frac{1}{4}; \tan \beta = 1. \]

We have \( \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - (\tan \alpha)(\tan \beta)} = \frac{4 + 1}{1 - (1)(\frac{1}{4})} = \frac{5}{3} \).

4. See Figure 4. Let \( A \) be the area of the triangle. Then using \( A = \frac{1}{2} \cdot bh \) with each of the three pairwise triangle side lengths and altitudes, we can express the triangle’s side lengths as \( \frac{2A}{3}, \frac{2A}{4}, \) and \( \frac{2A}{5} \). By Heron’s Formula, \( s = 1 \cdot \frac{2A}{3} + \frac{15A}{30} + \frac{12A}{30} = \frac{47A}{60} \), and
\[ A = \frac{\sqrt{\frac{47A}{30} \cdot \frac{17A}{60} \cdot \frac{23A}{60}}}{\frac{3600}{\sqrt{\frac{7\cdot 17 \cdot 23 \cdot 47}}}}. \]

Solve for \( A \):

5. See Figure 5. All points that are 2 units from the origin will lie on the circle \( x^2 + y^2 = 4 \). The lines we seek will be the tangents to this circle that pass through \( (2, 3) \). One such line is \( x = 2 \). Let the other line be tangent at the point \( (a, b) \). This line has slope \( \frac{3-b}{2-a} \), and the radius from the origin (with slope \( \frac{b}{a} \)) to the point of tangency is perpendicular to this. Thus \( \frac{3-b}{2-a} = -\frac{a}{b} \Rightarrow 3b-b^2 = 2a + a^2 \Rightarrow a^2 + b^2 = 3b - 2a \). But \( a^2 + b^2 = 4 \), so \( 3b - 2a = 4 \Rightarrow a = 2 - \frac{3}{4} b \). Substituting,
\[ \left( 2 - \frac{3}{4} b \right)^2 + b^2 = 4 \Rightarrow \frac{15}{4} b^2 - 6b = 0 \Rightarrow b = 0 \text{ or } \frac{24}{13} \Rightarrow (a, b) = \left( -\frac{10}{13}, \frac{24}{13} \right). \]

The second line is \( y - \frac{24}{13} = \frac{5}{12} \left( x + \frac{10}{13} \right) \).

6. \( f(x) \) is an upward-facing parabola whose minimum occurs at \( x = 20 \) and whose range includes the value \( y = 14 \). Therefore, the range of \( f(x) \) must include all \( y > 14 \) as well. In particular, \( 20 \) is in the range, and there must be two \( x \)-values, equally spaced about the axis of symmetry \( x = 20 \), for which \( f(x) = 20 \). Call these values \( 20 + d \) and \( 20 - d \). Note that since \( f(x) \) passes through \( (-3, 14) \), and \( x = -3 \) is 23 units away from \( x = 20 \), we can conclude that \( d > 23 \); that is, \( f(x) < 20 \) for \( -3 \leq x \leq 43 \).

Hence, the smallest positive integer \( x \) for which \( f(x) \) could equal 20, creating a minimum value for \( f(f(x)) \), is \( x = 44 \).
Quickies (8 points)

Questions in this section are intended to require very little computation and should be answered very quickly. Each question is worth 1 point. Place your answer to each question on the line provided.

1. Determine exactly the value of \( \sqrt[7]{64} \cdot \sqrt[14]{64} \cdot \sqrt[28]{64} \).

2. Alec glued together two right triangles with integer side lengths to form a third triangle, one whose sides measured 10, 17, and 21. Calculate the area of this triangle.

3. Determine exactly the value of \( x \) if \( \frac{4x-1}{2x+1} = \frac{2x-1}{x+5} \).

4. If \( \sqrt[2]{x+y} = 7 \) and \( \sqrt[3]{x-y} = 3 \), what is the value of \( \sqrt{x-y} \)?

5. The values of \( a, b, \) and \( c \) are such that the polynomial \( p(x) = (ax^2 + bx + c)(cx^2 + bx + a) \) has 4 real positive roots. Determine exactly the product of these roots.

6. The line \( \sqrt{3} \cdot x - 2y = 6 \) forms an acute angle with the positive \( x \)-axis. If the measure of this angle is \( \tan^{-1} n \), determine \( n \) exactly.

7. The roots of a quadratic equation are \( \frac{1}{3} \) and \( -\frac{2}{5} \). If the quadratic's coefficients are relatively prime integers and the coefficient of the \( x^2 \) term is positive, what is the sum of the coefficients?

8. If \( \frac{\pi}{2} < \theta < \pi \) and \( \sin \theta = \frac{\sqrt{5}}{3} \), determine \( \sin \frac{\theta}{2} \) exactly.

Name: ___________________________  Team: ___________________________
9. The lengths of the tangents from the vertices of a triangle to its inscribed circle are 4, 5, and 6. Determine exactly the area of this triangle.

\[ \text{Area} = \]  

10. Determine exactly the greatest possible value of \( c \) if \( a^2 + b^2 + c^2 = 6 \), \( a + b + c = 4 \), and \( a = 2b \).

\[ c = \]  

11. Find the least positive angle \( A \) (in degrees) for which \( \tan 37^\circ = \sqrt{\frac{1 + \sin A}{1 - \sin A}} \).

\[ m\angle A = \]  

12. Let \( \lfloor x \rfloor \) = the greatest integer less than or equal to \( x \).

Describe the set of all real numbers \( x \) such that \( 1 \leq x \lfloor x \rfloor \leq 6 \).

\[ \begin{align*}
13. & \quad \text{Determine the exact value of } \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 9^{n+1}}{4^{2n}}. \\
14. & \quad \text{Let } a \text{ and } b \text{ be real numbers. Determine exactly all values of the ratio } \frac{b}{a} \text{ such that } \\
& \quad \frac{ax+3}{bx+2} = \frac{ax-2}{bx-3} \text{ has no real solutions.} \\
& \quad b = \frac{a}{a} \\
15. & \quad \text{In } \triangle ABC, \text{ altitude } AD \text{ and sides } AB \text{ and } BC \text{ all have integer lengths.} \\
& \quad \text{If } AC = 53, \text{ find the smallest possible perimeter for } \triangle ABC. \\
& \quad \text{perimeter} = \\
\end{align*} \]

Name: ___________________________  Team: ___________________________
Quickies (8 points)

1. Determine exactly the value of \( \sqrt[3]{64} \cdot \sqrt[6]{64} \cdot \sqrt[20]{64} \cdot \sqrt[26]{64} \cdot \sqrt[14]{64} \cdot \sqrt[28]{64} = \frac{6}{14} \cdot \frac{6}{28} \cdot \frac{6}{2} = \frac{3}{2} \).

2. Alec glued together two right triangles with integer side lengths to form a third triangle, one whose sides measured 10, 17, and 21. Calculate the area of this triangle.

\[
\text{Treating 21 as the base, we have a 6-8-10 glued to an 8-15-17, with shared height 8. Area = } \frac{1}{2} \cdot 8 \cdot 21 = 84.
\]

3. Determine exactly the value of \( x \) if \( \frac{4x - 1}{2x + 1} = \frac{2x - 1}{x + 5} \).

Cross multiplying yields \( (4x - 1)(x + 5) = (2x + 1)(2x - 1) \Rightarrow 4x^2 + 19x - 5 = 4x^2 - 1 \Rightarrow x = \frac{4}{17} \).

4. If \( \sqrt{x + y} = 7 \) and \( \sqrt{x - y} = 3 \), what is the value of \( x - y \)?

\[
\sqrt{x + y} \cdot \sqrt{x - y} = \sqrt{(\sqrt{x})^2 - (\sqrt{y})^2} = \sqrt{x - y} = 7 \cdot 3 = 21.
\]

5. The values of \( a, b, \) and \( c \) are such that the polynomial \( p(x) = (ax^2 + bx + c)(cx^2 + bx + a) \) has 4 real positive roots. Determine exactly the product of these roots.

The product of the roots will be the constant term divided by the \( x^4 \) coefficient: \( \frac{ac}{ca} = 1 \).

6. The line \( \sqrt{3} \cdot x - 2y = 6 \) forms an acute angle with the positive \( x \)-axis.

If the measure of this angle is \( \tan^{-1} n \), determine \( n \) exactly.

The slope of the given line is \( \frac{\sqrt{3}}{2} \), which is equal to the desired tangent value.

7. The roots of a quadratic equation are \( \frac{1}{3} \) and \( \frac{-2}{5} \). If the quadratic's coefficients are relatively prime integers and the coefficient of the \( x^2 \) term is positive, what is the sum of the coefficients?

\( (3x - 1)(5x + 2) = 15x^2 + 1x - 2 \), whose coefficients sum to 14.

8. If \( \frac{\pi}{2} < \theta < \pi \) and \( \sin \theta = \frac{\sqrt{5}}{3} \), determine \( \sin \frac{\theta}{2} \) exactly.

\[
\cos \theta = \pm \sqrt{1 - \sin^2 \theta} = \pm \sqrt{1 - \left(\frac{5}{9}\right)} = \pm \frac{2}{3}. \quad \frac{\pi}{2} < \theta < \pi \Rightarrow \cos \theta = -\frac{2}{3}, \text{ so } \sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}} = \sqrt{\frac{5}{6}} = \frac{\sqrt{30}}{6}.
\]
9. The lengths of the tangents from the vertices of a triangle to its inscribed circle are 4, 5, and 6. Determine exactly the area of this triangle.

\[
\text{Area} = 30\sqrt{2}
\]

10. Determine exactly the greatest possible value of \( c \) if \( a^2 + b^2 + c^2 = 6, a+b+c = 4, \) and \( a = 2b. \)

\[
c = \frac{13}{7}
\]

11. Find the least positive angle \( A \) (in degrees) for which \( \tan 37^\circ = \frac{1 + \sin A}{1 - \sin A}. \)

\[
m\angle A = 196^\circ
\]

12. Let \( \left\lfloor x \right\rfloor \) = the greatest integer less than or equal to \( x. \)

Describe the set of all real numbers \( x \) such that \( 1 \leq x \left\lfloor x \right\rfloor \leq 6. \)

\[
\left[ -2, 1 \right] \cup \left[ 1, 3 \right)
\]

13. Determine the exact value of \( \sum_{n=1}^{\infty} \frac{(-1)^n 1}{4^{2n}} \cdot \left( \frac{9}{4} \right)^{n+1}. \)

\[
(2 \text{ pts})
\]

14. Let \( a \) and \( b \) be real numbers. Determine exactly all values of the ratio \( \frac{b}{a} \) such that \( \frac{ax+3}{bx+2} = \frac{ax-2}{bx-3} \) has no real solutions.

\[
b = \frac{2}{3} \quad a = \frac{3}{2}
\]

\[
(3 \text{ pts})
\]

15. In \( \triangle ABC \), altitude \( AD \) and sides \( AB \) and \( BC \) all have integer lengths. If \( AC = 53 \), find the smallest possible perimeter for \( \triangle ABC. \)

\[
108
\]

\[
(3 \text{ pts})
\]
9. See Figure 9. The side lengths of the circumscribed triangle are 9, 10, and 11. Using Heron’s formula, 
\[ \sqrt{15(15-9)(15-10)(15-11)} = \sqrt{15 \cdot 6 \cdot 5 \cdot 4} = \sqrt{4 \cdot 3^2 \cdot 5^2 \cdot 2} = 30\sqrt{2}. \]

10. \[ c = 4 - a - b = 4 - (2b) - b = 4 - 3b. \] By substitution, \( (2b)^2 + b^2 + (4 - 3b)^2 = 6 \Rightarrow 7b^2 - 12b + 5 = 0 \Rightarrow (7b - 5)(b - 1) = 0 \] \[ \Rightarrow b = \frac{5}{7} \text{ or } b = 1. \] The lesser of these \( b \) values will generate the greater \( c \) value: \( c = 4 - 3 \left( \frac{5}{7} \right) = \frac{13}{7}. \)

11. \[ \tan 37^\circ = \tan \left( \frac{74^\circ}{2} \right) = \sqrt{1 - \cos 74^\circ} \]. However, we would rather see sine values underneath that radical. By cofunction identity, \( \cos 74^\circ = \sin 16^\circ = -\sin (-16^\circ) \), and the least positive angle with that sine value is \( 180^\circ + 16^\circ = 196^\circ \).

12. For \( x \)-values larger than 0, \( x \) cannot be less than 1, because then \( \lfloor x \rfloor \) would be 0. We also may not use \( x \geq 3 \), because then \( \lfloor x \rfloor \) would yield values of at least 9. For \( x \)-values less than 0, \( x \) cannot be less than \(-2\), because the product \( \lfloor x \rfloor \) would yield values larger than 6. Furthermore, \(-1 < x < 0 \) yields products less than 1. Therefore, \( -2 \leq x \leq -1 \text{ or } 1 \leq x < 3 \).

13. \[ \sum_{n=1}^{\infty} \left( -\frac{1}{2} \right)^n \cdot \frac{9^{(n+1)}}{4^{2n}} \text{ as } -9 \sum_{n=1}^{\infty} \left( -\frac{9}{16} \right)^n \text{. This is a geometric series whose first term is } \frac{81}{16} \text{ and whose common ratio is } -\frac{9}{16}. \] The infinite sum of this series is then \( \frac{81/16}{1 - \left( -9/16 \right)} = \frac{81}{16} \cdot \frac{16}{25} = \frac{81}{25}. \)

14. First note that \( x \neq -\frac{2}{b} \text{ or } \frac{3}{b} \). \[ \frac{ax+3}{bx+2} = \frac{ax-2}{bx-3} \Rightarrow (ax + 3)(bx - 3) = (bx + 2)(ax - 2), \] which becomes \[ abx^2 - 3ax + 3bx - 9 = abx^2 - 2bx + 2ax - 4 \Rightarrow 5bx - 5ax = 5 \Rightarrow (b-a)x = 1. \] Thus, either \( b \neq a \left( \frac{b}{a} = 1 \right) \), or \( x = \frac{1}{b-a} \).

Because \( x \neq -\frac{2}{b} \), we set \( \frac{1}{b-a} = \frac{2}{b} \) and obtain \( b = 2a - 2b \Rightarrow 3b = 2a \Rightarrow \frac{b}{a} = \frac{2}{3} \). Similarly, because \( x \neq \frac{3}{b} \), we set \( \frac{1}{b-a} = \frac{3}{b} \Rightarrow b = 3a - 2b \Rightarrow \frac{b}{a} = \frac{3}{2}. \) Therefore, the ratios of \( \frac{b}{a} \) that give no solutions are \( \frac{1}{3}, \frac{2}{3}, \text{ and } \frac{3}{2}. \)

15. Note that \( \Delta ABC \) may be constructed either by building Pythagorean triangles on either side of \( \overrightarrow{AD} \) (making \( \Delta ABC \) acute), or by “nesting” one Pythagorean triangle inside the other, with both sharing “right leg” \( \overrightarrow{AD} \) (making \( \Delta ABC \) obtuse, and \( \overrightarrow{AD} \) an exterior altitude). Since we’re trying to minimize the perimeter, we probably will want to consider the “nesting” case, where 53 is the hypotenuse of both nested Pythagorean triangles. The only Pythagorean triple containing 53 as the hypotenuse is 28/45/53, so \( AD = 28 \) or 45. For either choice, look for Pythagorean triples that share that leg, and for which the other leg makes the perimeter as small as possible. If \( AD = 45 \), this triple is 24/45/51, which gives a triangle of side lengths 53, 51, and 4. If \( AD = 28 \), this triple is 21/28/35, which gives a triangle of side lengths 53, 35, 24. The smaller perimeter occurs in the first case, where \( P = 108 \).