Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 12 minutes for this event.

1. **Turkish liras**  
   Determine exactly how many Turkish liras 1 dollar will buy if a hotel room costs $54 for 81 Turkish liras.

2.  
   Determine the exact value of $\binom{7}{3}$.

3.  
   Let $a$ and $b$ be positive integers. If the greatest common factor of $a$ and $b$ is 10 and the least common multiple of $a$ and $b$ is 60, determine the minimum possible value of $a + b$.

4.  
   Given $S = \{1, 2, 3, \ldots, n-1, n\}$, determine the least value of $n$ such that there are exactly 1.25 times as many multiples of 19 in $S$ as multiples of 23.

Name: ___________________________  Team: ___________________________
1. Determine exactly how many Turkish liras 1 dollar will buy if a hotel room costs $54 for 81 Turkish liras.

<table>
<thead>
<tr>
<th>81 Turkish liras</th>
<th>9 Turkish liras</th>
<th>3 Turkish liras</th>
<th>1.5 Turkish liras</th>
</tr>
</thead>
<tbody>
<tr>
<td>$54</td>
<td>$6</td>
<td>$2</td>
<td>$1</td>
</tr>
</tbody>
</table>

\[
\frac{81 \text{ Turkish liras}}{54} = \frac{9 \text{ Turkish liras}}{6} = \frac{3 \text{ Turkish liras}}{2} = \frac{1.5 \text{ Turkish liras}}{1}
\]

\[
\begin{align*}
\frac{81}{54} & = \frac{9}{6} = \frac{3}{2} = \frac{1.5}{1} \\
\end{align*}
\]

min(\(a + b\)) = 50

2. Determine the exact value of \(\frac{.7}{.63}\).

\[
\frac{.7}{.63} = \frac{7}{63} = \frac{7}{9} \cdot \frac{7}{77} = \frac{7}{9} \cdot \frac{11}{11} = \frac{11}{9} = 1\frac{2}{9}
\]

3. Let \(a\) and \(b\) be positive integers. If the greatest common factor of \(a\) and \(b\) is 10 and the least common multiple of \(a\) and \(b\) is 60, determine the minimum possible value of \(a + b\).

Because \(a\) and \(b\) have a greatest common factor of 10, \(a\) and \(b\) must both be divisible by 10. This means the units digit of both \(a\) and \(b\) is 0. Since the LCM(a,b) = 60, \(a\) and \(b\) can each not be any larger than 60. Thus, the only values \(a\) or \(b\) can take are 10, 20, 30, or 60. Our possible ordered pairs for \((a,b)\) are \((10,60), (60,10), (20,30)\) and \((30,20)\), so \(a + b\) takes a minimum value of 50.

min\(n\) = 95

4. Given \(S = \{1, 2, 3, \ldots, n-1, n\}\), determine the least value of \(n\) such that there are exactly 1.25 times as many multiples of 19 in \(S\) as multiples of 23.

The multiples of 19 in set \(S\) are 19, 38, 57, 76, 95, 114, \ldots. The multiples of 23 in set \(S\) are 23, 46, 69, 92, 115, \ldots. We seek to find the least value \(n\), so try looking for exactly 5 multiples of 19 and 4 multiples of 23. Since \(5 \cdot 19 = 95\) and \(4 \cdot 23 = 92\), if \(n = 95\), there are exactly 5 multiples of 19 and 4 multiples of 23. Thus, \(n = 95\).
1. In Figure 1, if $\triangle ABC$ is equilateral and $CD$ is parallel to $AB$, calculate the measure of $\angle BCD$.

$m\angle BCD =$ __________

2. Town $A$ is located exactly 120 miles north of town $B$. If Sue hops in a car and drives directly east from town $B$ at 50 mph, calculate how many hours (as a decimal) it will take for Sue to be exactly 241 miles from town $A$ as the crow flies.

__________ hours

3. Right $\triangle ABC$ has $\overline{AD}$ congruent to $\overline{AB}$, as shown in Figure 3. If $BC = 7$ and $DC = 1$, determine exactly $AC$, the length of the hypotenuse.

$AC =$ __________

4. Three right triangles $ABC$, $DCE$, and $FEG$ are lined up in a row and mutually connected by $\overline{AG}$ to form one big right triangle $ABG$ (Figure 4). If $\angle BAC \cong \angle CDE \cong \angle DEC \cong \angle BCA$, $AG = 12$ and $m\angle BGA = 30^\circ$, determine exactly the length $FE$.

$FE =$ __________
**SOLUTIONS**

1. In Figure 1, if \( \triangle ABC \) is equilateral and \( CD \) is parallel to \( AB \), calculate the measure of \( \angle BCD \).

   \[ m\angle BCD = 60^\circ \]

   \[ m\angle ABC = m\angle BCD = 60^\circ \text{ since } \angle ABC \text{ and } \angle BCD \text{ are alternate interior angles and } \triangle ABC \text{ is equilateral.} \]

2. Town \( A \) is located exactly 120 miles north of town \( B \). If Sue hops in a car and drives directly east from town \( B \) at 50 mph, calculate how many hours (as a decimal) it will take for Sue to be exactly 241 miles from town \( A \) as the crow flies.

   \[ \text{Let } C \text{ equal Sue’s location when she is 241 miles from town } A. \text{ Form a right triangle between } A, B, \text{ and } C. \text{ Using the Pythagorean Theorem, we find } BC = \sqrt{241^2 - (120)^2} = \sqrt{43681} = 209. \]

   \[ \text{The time it takes for Sue to travel 209 miles is then } \frac{209 \text{ mi}}{50 \text{ mph}} = 4.18 \text{ hours.} \]

3. Right \( \triangle ABC \) has \( AD \) congruent to \( AB \), as shown in Figure 3. If \( BC = 7 \) and \( DC = 1 \), determine exactly \( AC \), the length of the hypotenuse.

   \[ \text{Let } AD = AB = x, \text{ as shown in Figure 3. By the Pythagorean Theorem,} \]

   \[ x^2 + 7^2 = (1 + x)^2 \Rightarrow x^2 + 49 = 1 + 2x + x^2 \Rightarrow 49 = 1 + 2x \Rightarrow x = 24, \]

   \[ \text{which means } AC = 1 + x = 25. \]

4. Three right triangles \( \triangle ABC \), \( \triangle DCE \), and \( \triangle FEG \) are lined up in a row and mutually connected by \( AG \) to form one big right triangle \( \triangle ABG \) (Figure 4). If \( \angle BAC \equiv \angle CDE \equiv \angle DEC \equiv \angle BCA \), \( AG = 12 \) and \( m\angle BGA = 30^\circ \), determine exactly the length \( FE \).

   \[ \text{ABC and DCE are both } 45^\circ - 45^\circ - 90^\circ \text{ right triangles and } AB = 6, \text{ so } BG = 6\sqrt{3}. \text{ CG is } 6\sqrt{3} - 6 \]

   \[ \text{because } BC = 6. \text{ Triangle DCG is a } 30^\circ - 60^\circ - 90^\circ \text{ right triangle, so } DC = 6 - 2\sqrt{3} \text{ and } CE = 6 - 2\sqrt{3}. \]

   \[ EG = \left(6\sqrt{3} - 6\right) - \left(6 - 2\sqrt{3}\right) = 8\sqrt{3} - 12. \text{ Therefore, } FE = \frac{8\sqrt{3} - 12}{\sqrt{3}} = 8 - 4\sqrt{3}. \]
1. Determine exactly the value of $\sin\frac{\pi}{3} - \cos 3\pi$.

2. If $\tan A = -\frac{\sqrt{39}}{5}$ and $\cos A = \frac{5}{8}$, determine exactly the value of $1 + \sin^2 A$.

3. Determine exactly the smallest angle $x > 0$ (in radians) for which the graphs $y = 6 \cos(7x + \pi)$ and $y = 3$ intersect.

4. Determine the number of angles $A$, where $A$ is an integer $0^\circ < A < 180^\circ$, for which $\cot A < \tan 111^\circ$. 

NO CALCULATORS are allowed on this event.
\( \frac{\sqrt{3}}{2} + 1 \) or \( \frac{\sqrt{3} + 2}{2} \)

1. Determine exactly the value of \( \sin \frac{\pi}{3} - \cos 3\pi \).

\[
\frac{\sqrt{3}}{2} - 1 = \frac{\sqrt{3} + 2}{2}.
\]

\[ \tan x = \frac{103}{64} \]

2. If \( \tan A = -\frac{\sqrt{39}}{5} \) and \( \cos A = \frac{5}{8} \), determine exactly the value of \( 1 + \sin^2 A \).

\[
\tan A = \frac{\sin A}{\cos A} \Rightarrow \sin A = \tan A \cdot \cos A = -\frac{\sqrt{39}}{5} \cdot \frac{5}{8} = -\frac{\sqrt{39}}{8}.
\]

Therefore, \( 1 + \sin^2 A = 1 + \left( -\frac{\sqrt{39}}{8} \right)^2 = \frac{103}{64} = 1 \frac{39}{64} \).

\[ \frac{2\pi}{21} \]

3. Determine exactly the smallest angle \( x > 0 \) (in radians) for which the graphs \( y = 6\cos(7x + \pi) \) and \( y = 3 \) intersect.

The graphs intersect when \( 6\cos(7x + \pi) = 3 \Rightarrow \cos(7x + \pi) = \frac{1}{2} \).

The cosine of an angle is \( .5 \) when the angle is equal to \( \frac{\pi}{3} + 2k\pi \) or \( \frac{5\pi}{3} + 2k\pi \), where \( k \) is an integer value, as shown in figure 3. The smallest positive angle occurs when \( 7x + \pi = \frac{5\pi}{3} \Rightarrow x = \frac{2\pi}{21} \).

\[ 20 \]

4. Determine the number of angles \( A \), where \( A \) is an integer \( 0^\circ < A < 180^\circ \), for which \( \cot A < \tan 111^\circ \).

111° is an angle in the second quadrant and since tangent values are negative in that quadrant, we can eliminate all angle values \( A \) in the first quadrant whose cotangents are positive. It is also important to note that 111° is an angle 21° from the positive y-axis. So \( \tan 111^\circ \) is the same as \( \cot(180^\circ - 21^\circ) = \cot 159^\circ \), see figure 4. As we choose angles closer to 180°, the cotangent values become more negative. Thus the angles for which \( \cot A < \tan 111^\circ \) are 160°, 161°, ..., 179°. There are 20 such angles.
**Minnesota State High School Mathematics League**

**2015-16 Meet 1, Individual Event D**

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

**NO CALCULATORS are allowed on this event.**

1. Let \( f(x) = x + 3 \) and \( g(x) = x^2 \). Determine exactly the value(s) of \( x \) for which \( g(f(x)) = 0 \).

\[ x = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \]

2. Find the remainder when \( 2x^3 - 9x^2 + 14x - 6 \) is divided by \( x + 2 \).

\[ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \]

3. The quadratic function \( f(x) = 3x^2 + kx + 5 \), where \( k \) is an integer, does not have any real roots. What is the greatest possible value of \( k \)?

\[ k = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \]

4. Let \( f(x) = 2x^2 - bx + 7 \) and \( g(x) = 2(x-c)^2 - 43 \).

If \( f(x) \) and \( g(x) \) are equal for all values of \( x \), determine exactly all possible values of \( b \).

\[ b = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \]
1. Let $f(x) = x + 3$ and $g(x) = x^2$. Determine exactly the value(s) of $x$ for which $g(f(x)) = 0$.

$$g(f(x)) = (x + 3)^2 = 0 \Rightarrow x = -3$$

2. Find the remainder when $2x^3 - 9x^2 + 14x - 6$ is divided by $x + 2$.

Using synthetic division we get

```
   -2 |  2   -9   14   -6
    -2
   ---|-----|-----|-----|-----
     2  -13  40  -86
```

$\Rightarrow$ remainder $= -86$. Another way to obtain the remainder is by using the Remainder Theorem to get

$$2(-2)^3 - 9(-2)^2 + 14(-2) - 6 = -86.$$ 

$k = 7$

3. The quadratic function $f(x) = 3x^2 + kx + 5$, where $k$ is an integer, does not have any real roots. What is the greatest possible value of $k$?

A quadratic function does not have any real roots when the value of its discriminant is less than 0. Therefore, $k^2 - 4 \cdot 3 \cdot 5 < 0 \Rightarrow k^2 < 60 \Rightarrow -\sqrt{60} < k < \sqrt{60}$. Since $k$ is the largest integer value for which this is true, $k = 7$.

$b = 20$

or $b = -20$

4. Let $f(x) = 2x^2 - bx + 7$ and $g(x) = 2(x - c)^2 - 43$.

If $f(x)$ and $g(x)$ are equal for all values of $x$, determine exactly all possible values of $b$.

Rewrite $g(x)$ in standard form to get $g(x) = 2x^2 - 4cx + 2c^2 - 43$. Equating the coefficients of each function we obtain $b = 4c$ and $2c^2 - 43 = 7$. Solving the second equation gives us $c$ values of $-5$ and $5$. This means $b = 20$ or $b = -20$. 

Graders: Award only 1 point if any extra solutions are given, or for only one correct solution.
1. If \( \overrightarrow{AB} + \overrightarrow{CD} = \frac{1}{3} \) with \( 0 < \overrightarrow{AB} < \overrightarrow{CD} \), determine exactly the sum of all pairs \( (\overrightarrow{AB}, \overrightarrow{CD}) \) satisfying those conditions.

2. In Figure 2, overlapping right triangles \( ABC \) and \( CDE \) are drawn such that \( AC = CD \) and \( ED \) is perpendicular to \( BC \). If \( AB = x \) and \( \angle BAC = \alpha \), express the length \( EF \) in terms of \( x \) and \( \alpha \) only.

3. Let \( g(x) \) be a linear function and let \( f(x) = x^2 - 6x + c \), where the roots of \( f(x) \) are \( 3 \pm 4i \). If the solutions to the equation \( f(x) = g(x) \) are \( x = 0 \) and the \( x \) coordinate of the axis of symmetry of \( f(x) \), write the formula for \( g(x) \).

4. A regular octagon \( MNOPQRST \) has sides of length 5, as shown in Figure 4. If sides \( PQ \) and \( ST \) are extended until they meet at point \( V \), find the length of \( TV \) exactly.

5. Determine exactly the value of \( k \), where \( k \) is positive, for which the polynomial \( 17x^2 - 19x + k \) has roots that are the sines of acute angles of some right triangle \( ABC \).

6. Let \( N = p^{2017} - 4p^{2016} + 4p^{2015} \), where \( N \) is a positive number. If \( p \) is a prime number, determine the least possible number of factors of \( N \).

Team: ________________________________
1. If $\overrightarrow{AB} + \overrightarrow{CD} = \frac{1}{3}$ with $0 < \overrightarrow{AB} < \overrightarrow{CD}$, determine exactly the sum of all pairs $(\overrightarrow{AB}, \overrightarrow{CD})$ satisfying those conditions.

\[
\begin{align*}
&\frac{16}{3} \\
&\frac{5}{3}
\end{align*}
\]

2. In Figure 2, overlapping right triangles $ABC$ and $CDE$ are drawn such that $AC = CD$ and $ED$ is perpendicular to $BC$. If $AB = x$ and $\angle BAC = \alpha$, express the length $EF$ in terms of $x$ and $\alpha$ only.

\[EF = \frac{x}{\tan \alpha}\]

or \[x \cot \alpha\]

3. Let $g(x)$ be a linear function and let $f(x) = x^2 - 6x + c$, where the roots of $f(x)$ are $3 \pm 4i$. If the solutions to the equation $f(x) = g(x)$ are $x = 0$ and the $x$ coordinate of the axis of symmetry of $f(x)$, write the formula for $g(x)$.

\[g(x) = -3x + 25\]

4. A regular octagon $MNOPQRST$ has sides of length 5, as shown in Figure 4. If sides $PQ$ and $ST$ are extended until they meet at point $V$, find the length of $TV$ exactly.

\[TV = 10 + 5\sqrt{2}\]

5. Determine exactly the value of $k$, where $k$ is positive, for which the polynomial $17x^2 - 19x + k$ has roots that are the sines of acute angles of some right triangle $ABC$.

\[k = \frac{36}{17}\]

or \[2 \frac{2}{17}\]

6. Let $N = p^{2017} - 4p^{2016} + 4p^{2015}$, where $N$ is a positive number.

If $p$ is a prime number, determine the least possible number of factors of $N$. 

\[2016\]
1. If \( N = \overline{AB} \), then \( 100N = \overline{AB} \) and subtraction by \( N \) gives \( 99N = 10A + B \). So, \( \overline{AB} = \frac{10A + B}{99} \) and, similarly, \( \overline{CD} = \frac{10C + D}{99} \). Thus, \( \frac{10(A + C) + (B + D)}{99} = \frac{33}{99} \) so \( 10(A + C) + (B + D) = 33 \). There are two possible cases that apply. The first is \( A + C = 3 \) and \( B + D = 3 \), then \( A = 0 \) and \( C = 3 \), in which \( (B,D) = (1,2), (2,1) \), and \( (3,0) \), or \( A = 1 \) and \( C = 2 \), in which \( (B,D) = (0,3), (1,2),(2,1) \), and \( (3,0) \). The second case occurs when \( A + C = 2 \) and \( B + D = 13 \). In this case \( A = 0 \) and \( C = 2 \), in which \( (B,D) = (4,9), (5,8), (6,7), (7,6), (8,5) \), and \( (9,4) \), or \( A = 1 \) and \( C = 1 \), in which \( (B,D) = (4,9), (5,8) \), and \( (6,7) \), since \( B \) must be less than \( D \). There are 16 possible ordered pairs for \( \overline{AB} \) and \( \overline{CD} \), each equal to \( \frac{1}{3} \), making the sum of all pairs \( \frac{16}{3} \).

2. Using standard right triangle trigonometry, \( \cos \alpha = \frac{x}{AC} \Rightarrow AC = \frac{x}{\cos \alpha} \). Since \( AC = CD \) and \( m \angle BAC = m \angle FCD = \alpha \), the right triangles \( BAC \) and \( FCD \) are congruent and \( FC = x \). Using right triangle trigonometry once more, we find

\[
\tan \angle FEC = \tan \alpha = \frac{x}{EF} \Rightarrow EF = \frac{x}{\tan \alpha}.
\]

3. If \( f(x) \) has the roots \( 3 \pm 4i \), then \( c \) is the product of the roots, since the \( x^2 \) coefficient of \( f(x) \) is 1. Therefore, \( c = (3 + 4i)(3 - 4i) = 25 \), \( f(x) = x^2 - 6x + 25 \). \( g(x) \) is a linear equation and can be written as \( g(x) = mx + b \). The equation \( f(x) = g(x) \Rightarrow x^2 - 6x + 25 = mx + b \Rightarrow x^2 -(6+m)x+25-b=0 \). Since 0 is a solution to the equation \( f(x) = g(x) \), then \( (0)^2 -(6+m)(0)+25-b = 0 \Rightarrow 25 - b = 0 \Rightarrow b = 25 \). The \( x \) value of the axis of symmetry of \( f(x) \) is \( \frac{-(-6)}{2(1)} = 3 \), which is a solution to the equation \( f(x) = g(x) \). This means \( (3)^2 -(6+m)(3) = 0 \Rightarrow 9 - 18 - 3m = 0 \Rightarrow m = -3 \). Therefore, \( g(x) = -3x + 25 \).

4. See Figure 4. Draw \( \overline{SP} \) and perpendicular segments to \( \overline{SP} \) from vertices \( R \) and \( Q \), creating right triangles \( SKR \) and \( PLQ \).

Using \( 45^\circ - 45^\circ - 90^\circ \) \( \triangle SKR \), we have \( SK = KR = \frac{5\sqrt{2}}{2} \). Similarly, \( \triangle PLQ \) is also a \( 45^\circ - 45^\circ - 90^\circ \) triangle, and \( PL = LQ = \frac{5\sqrt{2}}{2} \).

So \( SP = SK + KL + LP = 5 + 5\sqrt{2} \). \( \triangle SV \) is also a \( 45^\circ - 45^\circ - 90^\circ \) triangle, making \( SV = 5 + 5\sqrt{2} \). So, \( TV = SV = \frac{10 + 5\sqrt{2}}{2} \).

5. Let the acute angles be \( A \) and \( B \). Since \( A + B = 90^\circ \), \( \sin B = \sin(90^\circ - A) = \cos A \). If we call the roots \( r = \sin A \) and \( s = \cos A \), we know \( r^2 + s^2 = 1. \) The equation \( 17x^2 - 19x + k = 0 \Rightarrow x^2 = \frac{19}{17} - \frac{k}{17} = 0 \), which means \( r + s = \frac{19}{17} \) and \( rs = \frac{k}{17} \). Since

\[
r^2 + s^2 = (r + s)^2 - 2rs = \left(\frac{19}{17}\right)^2 - 2\left(\frac{k}{17}\right) = 1 = \frac{361}{289} - \frac{2k}{17}.
\]

Solving for \( k \), we get \( k = \frac{36}{17} - 2\left(\frac{2}{17}\right) \).

6. Rewrite \( N \) as \( N = p^{2017} - 4p^{2016} + 4p^{2015} = p^{2015}(p^2 - 4p + 4) = p^{2015}(p - 2)^2 \). The value of \( p \) that will give us the fewest number of factors is 3. This means that \( N = 3^{2015}(3 - 2)^2 = 3^{2015} \), which has \( 2016 \) factors.