1. A woman invests her money and notices it doubles every three years when compounded annually at r %. In how many years will she have eight times her original investment in this account?

2. We call a date *swell* if the number of the month is a factor of the day of that month. For example, February 14 is *swell* since 2 is a factor of 14, but December 25 is not *swell* since 12 is not a factor of 25. If a day is selected randomly during the year (assuming it is not a leap year), what is the probability the day is *swell*?

3. Determine exactly the ordered quadruple \((w, x, y, z)\) which satisfies this system:

\[
\begin{align*}
2w + x + y + z &= 5 \\
w + 2x + y + z &= 10 \\
w + x + 2y + z &= 20 \\
w + x + y + 2z &= 40
\end{align*}
\]

4. The solutions for \(x\) to \((k)(x - 2) = \frac{1}{1 + \frac{1}{x - 2}}\) that lie between 9 and 19, inclusive, generate values of \(k\) in the interval \([a, b]\). Determine exactly \([a, b]\), where \(a < b\).
1. A woman invests her money and notices it doubles every three years when compounded annually at \( r \% \). In how many years will she have eight times her original investment in this account?

\[
2P = P\left(1 + \frac{r}{100}\right)^3 \Rightarrow 2 = \left(1 + \frac{r}{100}\right) \Rightarrow 2^3 = \left(1 + \frac{r}{100}\right)^3 \Rightarrow 8P = P\left(1 + \frac{r}{100}\right)^9.
\]

\[
\text{or} \quad \frac{90}{365} \quad \text{or} \quad \frac{18}{73}
\]

2. We call a date swell if the number of the month is a factor of the day of that month. For example, February 14 is swell since 2 is a factor of 14, but December 25 is not swell since 12 is not a factor of 25. If a day is selected randomly during the year (assuming it is not a leap year), what is the probability the day is swell?

Just count the number of swell days each month: \(31+14+10+7+6+5+4+3+3+2+2 = 90\).

3. Determine exactly the ordered quadruple \((w, x, y, z)\) which satisfies this system:

\[
\begin{align*}
2w + x + y + z &= 5 \\
w + 2x + y + z &= 10 \\
w + x + 2y + z &= 20 \\
w + x + y + 2z &= 40
\end{align*}
\]

Adding the equations together produces: \(5w + 5x + 5y + 5z = 5(1 + 2 + 4 + 8)\).

Therefore, \(w + x + y + z = 15\). Subtracting this equation from each of the equations in the system produces: \(w = -10\), \(x = -5\), \(y = 5\), and \(z = 25\).

\[
\begin{bmatrix}
1 & 1 \\
18 & 8
\end{bmatrix}
\]

or

\[
[0.05, 0.125]
\]

4. The solutions for \(x\) to \((k)(x - 2) = \frac{1}{1 + \frac{1}{x - 2}}\) that lie between 9 and 19, inclusive, generate values of \(k\) in the interval \([a, b]\). Determine exactly \([a, b]\), where \(a < b\).

\[
k(x - 2) = \frac{x - 2}{x - 1} \Rightarrow k = \frac{1}{x - 1} \Rightarrow x = \frac{1}{k} + 1. \text{ Then } 9 \leq \frac{1}{k} + 1 \leq 19 \Rightarrow 8 \leq \frac{1}{k} \leq 18 \Rightarrow \frac{1}{18} \leq k \leq \frac{1}{8}.
\]
Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this Tournament event.

1. In Figure 1, ΔABC is equilateral. Segments AW, AX, AY, and AZ divide ∠BAC into five equal angles. Determine exactly the degree measure of ∠AXB.

2. ABC is an isosceles triangle with AB ≅ AC. Point D is on BC such that the distance from D to AB is 8 and the distance from D to AC is 16. If BC = 30, what is the area of ΔABC?

3. Right triangle ABC has legs AB and BC of lengths 20 and 21, respectively. M is the midpoint of AB and N is the trisection point of BC closest to C. If AN and CM intersect at O and ray BO intersects AC at P, determine exactly the area of ΔABP.

4. In ΔABC, points D and E lie on AB and AC, respectively, such that DE || BC. If [ADE] = 20 and [BDE] = 18, determine exactly [ABC].
1. In Figure 1, \( \triangle ABC \) is equilateral. Segments \( \overline{AW}, \overline{AX}, \overline{AY}, \) and \( \overline{AZ} \) divide \( \angle BAC \) into five equal angles. Determine exactly the degree measure of \( \angle AXB \).

\[ \angle BAC = 60^{\circ}, \text{ so } \angle BAX = \frac{1}{5}60^{\circ} = 24^{\circ}, \text{ and } \angle AXB = 180^{\circ} - 24^{\circ} - 60^{\circ} = 96^{\circ}. \]

2. \( ABC \) is an isosceles triangle with \( \overline{AB} \cong \overline{AC} \). Point \( D \) is on \( \overline{BC} \) such that the distance from \( D \) to \( \overline{AB} \) is 8 and the distance from \( D \) to \( \overline{AC} \) is 16. If \( BC = 30 \), what is the area of \( \triangle ABC \)?

See Figure 2.1. Let \( G \) be the midpoint of \( \overline{BC} \). Then \( \triangle AGC \sim \triangle DFC \sim \triangle DEB \). Since \( DF \) is twice \( DE \), \( DC \) is twice \( DB \). So \( DB=10 \) and \( DC=20 \). By the Pythagorean Theorem, \( FC=12 \).

\[ AG = \frac{DE}{FC} \Rightarrow AG = \frac{16}{12} \Rightarrow AG = 20. \text{ } [ABC] = \frac{1}{2}(30)(20) = 300. \]

3. Right triangle \( ABC \) has legs \( \overline{AB} \) and \( \overline{BC} \) of lengths 20 and 21, respectively. \( M \) is the midpoint of \( \overline{AB} \) and \( N \) is the trisection point of \( \overline{BC} \) closest to \( C \). If \( \overline{AN} \) and \( \overline{CM} \) intersect at \( O \) and ray \( \overline{BO} \) intersects \( \overline{AC} \) at \( P \), determine exactly the area of \( \triangle ABP \).

By the Pythagorean Theorem, \( AC = 29 \). Let \( AP = x \). By Ceva’s Theorem, \( \left( \frac{1}{1} \right) \left( \frac{2}{1} \right) \left( \frac{29-x}{x} \right) = 1 \Rightarrow \)

\[ \frac{29-x}{x} = \frac{1}{2} \Rightarrow x = \frac{58}{3}. \text{ Let } \overline{BO} = h \text{ be the height of } \triangle ABP. \text{ } \triangle AQB \sim \triangle ABC \Rightarrow \]

\[ h = \frac{AB}{AC} \Rightarrow h = \frac{21 \cdot 20}{29} = \frac{420}{29}. \text{ Therefore, } [ABP] = \frac{1}{2} \left( \frac{58}{3} \right) \left( \frac{420}{29} \right) = 140. \]

4. In \( \triangle ABC \), points \( D \) and \( E \) lie on \( \overline{AB} \) and \( \overline{AC} \), respectively, such that \( \overline{DE} \parallel \overline{BC} \).

If \([ADE] = 20\) and \([BDE] = 18\), determine exactly \([ABC]\).

\( \triangle ADE \) and \( \triangle BDE \) have the same height to bases \( \overline{AD} \) and \( \overline{BD} \). So \([ADE] = \frac{AD}{BD} = \frac{20}{18} = \frac{10}{9}. \text{ Therefore, } \frac{AB}{AD} = \frac{19}{10}, \text{ and } [ABC] = \left( \frac{19}{10} \right)^2 \cdot 20 = \frac{361}{5}. \)
1. If \( \tan x = \frac{2ab}{a^2 - b^2} \), where \( a > b > 0 \) and \( 0^\circ < x < 90^\circ \), what is \( \sin x \) in terms of \( a \) and \( b \)?

\[
\quad
\]

2. For \( 0 \leq x, y < 2\pi \), determine exactly the two ordered pairs \((x, y)\) that satisfy this system of equations:

\[
\sqrt{2} \cos x = 1 + \cos y \\
\sqrt{2} \sin x = \sin y
\]

\[
\quad
\]

3. Lines \( l_1 \) and \( l_2 \) are parallel. Five points, \( A, B, C, D \) and \( E \), are on \( l_1 \), and ten points, \( Q, R, S, T, U, V, W, X, Y \), and \( Z \), are on \( l_2 \). Using these fifteen points, how many distinct triangles are determined?

\[
\quad
\]

4. Determine exactly the four solutions to this equation in the interval \( [0, 2\pi) \):

\[
1 + \sin(2x) = 3\cos(2x).
\]

\[
\quad
\]
NO CALCULATORS are allowed on this event.

1. If \( \tan x = \frac{2ab}{a^2 - b^2} \), where \( a > b > 0 \) and \( 0^\circ < x < 90^\circ \), what is \( \sin x \) in terms of \( a \) and \( b \)?

   **Proof without words:**

2. For \( 0 \leq x, y < 2\pi \), determine exactly the two ordered pairs \( (x, y) \) that satisfy this system of equations:
   \[
   \begin{align*}
   \sqrt{2} \cos x &= 1 + \cos y \\
   \sqrt{2} \sin x &= \sin y
   \end{align*}
   \]

   **Squaring yields:** \( 2\cos^2 x = 1 + 2 \cos y + \cos^2 y \) and \( 2\sin^2 x = \sin^2 y \). **Adding the equations gives:**

   \[
   2\left(\sin^2 x + \cos^2 x\right) = 1 + 2 \cos y + \cos^2 y + \sin^2 y = 2 = 2 + 2 \cos y \Rightarrow 2 \cos y = 0 \Rightarrow y = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}.
   \]

   **Letting** \( y = \frac{\pi}{2} \) **yields** \( x = \frac{\pi}{4} \) **and letting** \( y = \frac{3\pi}{2} \), **yields** \( x = \frac{7\pi}{4} \).

3. Lines \( l_1 \) and \( l_2 \) are parallel. Five points, \( A, B, C, D \) and \( E \), are on \( l_1 \), and ten points, \( Q, R, S, T, U, V, W, X, Y, \) and \( Z \), are on \( l_2 \). Using these fifteen points, how many distinct triangles are determined?

   **Select two points from one line and one point from the other line:**

   \[
   \begin{pmatrix} 5 \\ 2 \end{pmatrix} \begin{pmatrix} 10 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix} \begin{pmatrix} 10 \\ 2 \end{pmatrix} = 10 \cdot 10 + 5 \cdot 45 = 325.
   \]

4. Determine exactly the four solutions to this equation in the interval \( [0, 2\pi] \):

   \( 1 + \sin(2x) = 3\cos(2x) \).

   **GRADERS:** \( \tan^{-1} \left( \frac{1}{2} \right) \) **could be replaced by** \( \sin^{-1} \left( \frac{\sqrt{2}}{4} \right) \) **or** \( \cos^{-1} \left( \frac{\sqrt{2}}{4} \right) \).

   Simplifying:

   \[
   1 + 2 \sin x \cos x = 3\cos^2 x - 3\sin^2 x \Rightarrow \frac{1}{\cos^2 x} + 2\tan x = 3 - 3\tan^2 x
   \]

   \[
   \tan^2 x + 1 + 2 \tan x = 3 - 3\tan^2 x \Rightarrow 4\tan^2 x + 2\tan x - 2 = 0 \Rightarrow 2(\tan x - 1)(\tan x + 1) = 0.
   \]

   Therefore, \( \tan x = \frac{1}{2} \) or \( \tan x = -1 \). So \( x = \tan^{-1} \left( \frac{1}{2} \right), \ x = \pi + \tan^{-1} \left( \frac{1}{2} \right), \ x = \frac{3\pi}{4}, \) or \( x = \frac{7\pi}{4} \).
Minnesota State High School Mathematics League
2018-19 State Tournament, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this Tournament event.

NO CALCULATORS are allowed on this event.

1. Three parallel lines $l_1$, $l_2$, and $l_3$ intersect with transversal $l_4$. Lines $l_1$ and $l_4$ intersect at $(4, 9)$, lines $l_2$ and $l_4$ intersect at $(8, 14)$, and lines $l_3$ and $l_4$ intersect at $(9, y)$. Determine exactly the value of $y$. 

$r = \text{__________} \hspace{0.5cm} 2. \text{ Determine exactly the radius of the circle that passes through the origin and is concentric with the circle } x^2 + y^2 + 6x - 14y + 54 = 0.$

$\text{__________} \hspace{0.5cm} 3. \text{ Let } ABCDEF \text{ be a regular hexagon with side length } \sqrt{3}. \text{ Let } X, Y \text{ and } Z \text{ be the midpoints of } AB, CD, \text{ and } EF, \text{ respectively. Determine exactly the area of the convex hexagon whose interior is the intersection of } \triangle ACE \text{ and } \triangle XYZ.$

$\text{__________} \hspace{0.5cm} 4. \text{ Determine exactly the value of } x \text{ such that } 7^{\log x} = 98 - x^{\log 7}.$
1. Three parallel lines \( l_1, l_2, \) and \( l_3 \) intersect with transversal \( l_4 \). Lines \( l_1 \) and \( l_4 \) intersect at \((4, 9)\), lines \( l_2 \) and \( l_4 \) intersect at \((8, 14)\), and lines \( l_3 \) and \( l_4 \) intersect at \((9, y)\).

Determine exactly the value of \( y \).

Using the two given points, the equation for \( l_4 \) is \( y = \frac{5}{4} x + 4 \). Letting \( x = 9 \), \( y = \frac{61}{4} \).

2. Determine exactly the radius of the circle that passes through the origin and is concentric with the circle \( x^2 + y^2 + 6x - 14y + 54 = 0 \).

Completing the squares: \((x+3)^2 + (y-7)^2 = 4\). A circle concentric to this circle has the equation \((x+3)^2 + (y-7)^2 = r^2 \Rightarrow (0+3)^2 + (0-7)^2 = r^2 \Rightarrow 9 + 49 = r^2 \Rightarrow r = \sqrt{58} \).

3. Let \( ABCDEF \) be a regular hexagon with side length \( \sqrt{3} \). Let \( X, Y \) and \( Z \) be the midpoints of \( AB, CD, \) and \( EF \), respectively. Determine exactly the area of the convex hexagon whose interior is the intersection of \( \triangle ACE \) and \( \triangle XYZ \).

Because of symmetry, all right triangles in the diagram are \( 30^\circ\cdot 60^\circ\cdot 90^\circ \). If \( ED = \sqrt{3} \), then \( OD = \frac{\sqrt{3}}{2} \), and \( OE = \frac{3}{2} \). Therefore, \( CE = 3 \) and \( [ACE] = \frac{3\sqrt{3}}{4} \). \( \triangle ACE \) is divided into 8 congruent right triangles. Hexagon \( LMNOPQ \) includes five of these triangles.

Therefore, \( [LMNOPQ] = \frac{5}{8} \cdot \frac{9\sqrt{3}}{4} = \frac{45\sqrt{3}}{32} \).

4. Determine exactly the value of \( x \) such that \( 7^{\log x} = 98 - x^{\log 7} \).

Use change of base law: \( \frac{\log x}{\log 10} + \frac{\log 7}{\log 10} = 98 \). Letting \( a = \log_{10} 7 \) yields \( \frac{\log x}{a} + \frac{1}{a} = 98 \Rightarrow \left(7^{\log x}\right)^{\frac{1}{a}} + x^{\frac{1}{a}} = 98 \Rightarrow x^a + x^a = 98 \Rightarrow 2\left(x^a\right)^{\frac{1}{a}} = 98 \Rightarrow x^a = 49 \Rightarrow x^{\log_{10} 7} = 49 \). Therefore, \( \log_{10} 7 = \log x \Rightarrow \log_{10} 7 = 2 \log x \Rightarrow \log_{10} 7 = 2 \frac{\log 7}{\log_{10} x} \Rightarrow \log x = 2 \Rightarrow x = 100 \).
1. In how many distinct ways can you place the nine 5’s on a Sudoku grid in Figure 1? (There must be one 5 in every 3 by 3 block, one 5 in every row, and one 5 in every column.)

(Note: Rotations and reflections of a completed grid are considered distinct.)

2. The base 10 number $N$ has three digits when written in base 7. When $N$ is expressed in base 9, the digits are reversed. What is the base 10 representation for $N$?

3. A and B travel around a circular track at uniform speeds in opposite directions and starting from diametrically opposite points on the circle. If they start at the same time, meet first after B has traveled 100 yds., and meet a second time 60 yds. before A completes a lap, determine exactly the circumference of the track.

4. When 9638, 8739, and 2591 are divided by integer $D$, where $D > 1$, the remainder is the same for all three divisions. Determine the value of $D$.

5. For $n > 2$, Figure 5 shows part of a sequence of $n$ angles that sum up to 143°. If $x$ is a positive integer, what is the least possible value for $x$?

6. A sphere of radius $R$ is inscribed in a cone. Another sphere is circumscribed about the cone. If the centers of the two spheres coincide, what is the volume of the cone in terms of $R$?

Team: ______________________________
1. In how many distinct ways can you place the nine 5’s on a Sudoku grid in Figure 1? (There must be one 5 in every 3 by 3 block, one 5 in every row, and one 5 in every column.)
(Note: Rotations and reflections of a completed grid are considered distinct.) Figure 1.1

2. The base 10 number $N$ has three digits when written in base 7. When $N$ is expressed in base 9, the digits are reversed. What is the base 10 representation for $N$?

3. A and B travel around a circular track at uniform speeds in opposite directions and starting from diametrically opposite points on the circle. If they start at the same time, meet first after B has traveled 100 yds., and meet a second time 60 yds. before A completes a lap, determine exactly the circumference of the track.

4. When 9638, 8739, and 2591 are divided by integer $D$, where $D > 1$, the remainder is the same for all three divisions. Determine the value of $D$.

5. For $n > 2$, Figure 5 shows part of a sequence of $n$ angles that sum up to 143°. If $x$ is a positive integer, what is the least possible value for $x$?

6. A sphere of radius $R$ is inscribed in a cone. Another sphere is circumscribed about the cone. If the centers of the two spheres coincide, what is the volume of the cone in terms of $R$?
1. There are 9 ways to place a 5 in Block A. Place it in a cell and cross out the row and column containing that cell. There are now 6 ways to place a 5 in Block B. Place it in a cell and cross out the row and column containing that cell. There are now 3 ways to place a 5 in Block C. Place it in a cell and cross out the row and column containing that cell. There are now 5 ways to place a 5 in Block D. Place it in a cell and cross out the row and column containing that cell. There are now 4 ways to place a 5 in Block E. Place it in a cell and cross out the row and column containing that cell. There are now 2 ways to place a 5 in Block F. Place it in a cell and cross out the row and column containing that cell. There are now 4 ways to place a 5 in Block G. Place it in a cell and cross out the row and column containing that cell. There are now 2 ways to place a 5 in Block H. Place it in a cell and cross out the row and column containing that cell. There is now only one cell open in Block I to place the last 5. Total number of ways: \(9 \cdot 6 \cdot 3 \cdot 6 \cdot 4 \cdot 2 \cdot 3 \cdot 2 \cdot 1 = 3^6 \cdot 2^6 = 6^6 = 46656\).

2. Let \(N = xyzn\), and \(N = zyxn\). Then \(49x + 7y + z = 81z + 9y + x \Rightarrow 48x - 80z = 2y \Rightarrow 24x - 40z = y \Rightarrow 8(3x - 5z) = y\). Since \(y < 7\), \(3x - 5z = 0 \Rightarrow y = 0\) and \(3x = 5z \Rightarrow x = 5\) and \(z = 3\). So \(N = 503\), and \(N = 305_9\). Therefore, \(N = 5 \cdot 49 + 3 = 3 \cdot 81 + 5 = 248_{10}\).

3. Let \(a = A's\) speed, \(b = B's\) speed, and \(2c = the\ circumference\).

\[
\frac{100}{b} = \frac{c - 100}{a} \Rightarrow \frac{a}{b} = \frac{c - 100}{100} \quad \text{and} \quad \frac{2c - 60}{a} = \frac{c + 60}{b} \Rightarrow \frac{2c - 60}{c + 60} = \frac{a}{b}.
\]

Therefore,

\[
\frac{c - 100}{100} = \frac{2c - 60}{c + 60} \Rightarrow c^2 - 40c - 6000 = 200c - 6000 \Rightarrow c^2 = 240c \Rightarrow c = 240 \Rightarrow 2c = 480.
\]

4. \(9638 = a \cdot D + R, 8739 = b \cdot D + R, \) and \(2591 = c \cdot D + R\). So \(9638 - 8739 = (a - b) \cdot D = 899 = 29 \cdot 31\). \(9638 - 2591 = (a - c) \cdot D = 7047 = 3^5 \cdot 29\). \(8739 - 2591 = (b - c) \cdot D = 6148 = 2^3 \cdot 29 \cdot 53\). Since 29 is the only common factor of all three differences greater than 1, \(D = 29\).

5. The measure of the \(n\)th angle is \(x + (n - 1)\) and the sum of the measures of these \(n\) angles is

\[
(x + x + (n - 1)) \left(\frac{n}{2}\right) = 143 \Rightarrow n(2x + n - 1) = 286 = 2 \cdot 11 \cdot 13.\]

Therefore, there are several possibilities for \(n\): 11, 13, 22, 26, 143, and 286. If \(n = 11\), \(2x + 11 - 1 = 26 \Rightarrow x = 8\). If \(n = 13\), \(2x + 13 - 1 = 22 \Rightarrow x = 5\). If \(n = 22\), \(2x + 22 - 1 = 13 \Rightarrow x = -4\) (impossible). All the other possibilities for \(n\) also produce a negative value for \(x\). So the least positive value for \(x\) is 5.

6. Consider the cross-section formed when the spheres and cone are cut in half. (See Figure 6.) Since the incenter and circumcenter coincide, the cross-section of the cone is an equilateral triangle. The volume of the cone is \(\frac{1}{3} \pi \left(R \sqrt{3}\right)^2 (3R) = 3 \pi R^3\).

\[\text{Figure 6}\]
Questions in this section are intended to require very little computation and should be answered very quickly. Each question is worth 1 point. Place your answer to each question on the line provided.

1. Calculate the value of $\sqrt{lcm(144,36) \cdot gcf(144,36)}$.

2. If a team had won three more of the games they played, its percent of games won would have increased by exactly 2.40. How many games had they played?

3. Let $f(x) = x^3$ and $g(x) = x - 7$. Determine exactly the value of $f(g(16)) - g(f(16))$.

4. Simplify the expression $\sqrt{3 + \sqrt{5} + \sqrt{3 - \sqrt{5}}}$ so that it no longer involves nested radicals.

5. $ABC$ is an isosceles right triangle with hypotenuse $BC$. Point $D$ is chosen such that $\angle BDC$ is right and $\angle BCD = 30^\circ$. Determine exactly the ratio $CD : AB$.

6. A right triangle with a perimeter of 60 has sides whose lengths form an arithmetic progression. Determine exactly its area.

7. Each of a group of 50 girls is either blonde or brunette and each has blue eyes or brown eyes. If in the group there are 18 girls with brown eyes, 31 girls are brunettes, and 14 girls are blue-eyed blondes, how many brown-eyed brunettes are in the group?

8. Let $S = 2 + 4 + 6 + \ldots + 2N$, where $N$ is the smallest positive integer such that $S > 1,000,000$. Determine exactly the value of $N$.

Name: _______________________________ Team: _______________________________
9. In isosceles triangles $ABC$ and $PQR$, $\angle A \cong \angle P$, $AB = PQ = x$, and $BC = QR = 1$, but the two triangles are NOT similar. If $x > 1$, determine exactly the value of $x$.

10. If $a > 0$ and $(a, b)$ is a solution to the system

$$|x + 2y| = 6 \quad \text{and} \quad |x| - 4|y| = 3,$$

determine exactly the greatest possible value for $a + b$.

11. $z^{2019} = 1$ has 2019 solutions in the complex plane. Calculate the product of all 2019 solutions. Express your answer in the form $rcis\, \theta$, where $0 \leq \theta < 2\pi$ and $r > 0$.

12. Compute the number of ordered pairs of positive integers $(x, y)$ for which $3x + 5y = 2019$.

13. There is only one four-digit integer of the form $aabb$ that is a perfect square. Find it.

14. $\triangle ABC$ has a right angle at $C$. Squares $ABDE$, $CAFG$, and $BCHI$ are constructed as in Figure 14. If $DI = 6$ and $EF = 7$, determine exactly $HG$.

---

Name: ___________________________   Team: ___________________________
1. Calculate the value of |\( \sqrt{\text{lcm}(144, 36) \cdot \text{gcf}(144, 36)} \)|.

|\( \text{lcm}(m, n) \cdot \text{gcf}(m, n) = m \cdot n. \text{Therefore, the expression is equivalent to} \sqrt{144 \cdot 36} = 12 \cdot 6 = 72. \)

2. If a team had won three more of the games they played, its percent of games won would have increased by exactly 2.40. How many games had they played?

|Let \( x = \# \text{ of victories} \) and \( y = \# \text{ of games. Then} \)|
|\( \frac{x + 3}{y} = 100 \Rightarrow 100x + 300 = 100x + 2.4y \Rightarrow y = \frac{300}{2.4} = 125. \)

3. Let \( f(x) = x^{3/2} \) and \( g(x) = x - 7. \text{ Determine exactly the value of} f(g(16)) - g(f(16)). \)

|\( g(16) = 9 \) and \( f(16) = 64. f(9) = 27 \) and \( g(64) = 57. \text{ Therefore,} f(g(16)) - g(f(16)) = 27 - 57 = -30. \)

4. Simplify the expression \( \sqrt{3 + \sqrt{5}} + \sqrt{3 - \sqrt{5}} \) so that it no longer involves nested radicals.

|\( (\sqrt{3 + \sqrt{5}} + \sqrt{3 - \sqrt{5}})^2 = 3 + \sqrt{5} + 2(\sqrt{3 + \sqrt{5}})(\sqrt{3 - \sqrt{5}}) + 3 - \sqrt{5} = 6 + 2\sqrt{4} = 10. \)

5. \( ABC \) is an isosceles right triangle with hypotenuse \( BC. \) Point \( D \) is chosen such that \( \angle BDC \) is right and \( \angle BCD = 30^\circ. \) Determine exactly the ratio \( CD : AB. \)

|\( \frac{\sqrt{6}}{2} : 1 \) or \( \frac{\sqrt{6}}{2} \)\n\( \frac{\sqrt{6}}{2} : \frac{\sqrt{3}}{\sqrt{2}} \) or \( \sqrt{3} : \sqrt{2} \)|

|\( \text{Let} AC = AB = 1, \text{ then} BC = \sqrt{2}, BD = \frac{\sqrt{2}}{2}, \text{ and} DC = \sqrt{3} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{2}. \)

6. A right triangle with a perimeter of 60 has sides whose lengths form an arithmetic progression. Determine exactly its area.

|\( \text{Let the sides be} a - d, a, \text{ and} a + d. \text{ Then} a - d + a + a + d = 60 \Rightarrow a = 20. \text{ Then} (20 - d)^2 + 20^2 = (20 + d)^2 \Rightarrow 400 - 40d + d^2 + 400 = 400 + 40d + d^2 \Rightarrow 800 = 80d \Rightarrow d = 5. \text{ Sides are 15, 20, 25 and area =} \frac{1}{2}(20)(15) = 150. \)

7. Each of a group of 50 girls is either blonde or brunette and each has blue eyes or brown eyes. If in the group there are 18 girls with brown eyes, 31 girls are brunettes, and 14 girls are blue-eyed blondes, how many brown-eyed brunettes are in the group?

|\( \text{Let} x = \# \text{ of brown-eyed brunettes} \text{ and make a Vietch Diagram:} \)|
|\( 14 + 18 - x + 31 - x + x = 50 \Rightarrow 63 - x = 50 \Rightarrow x = 13. \)

8. Let \( S = 2 + 4 + 6 + \ldots + 2N, \) where \( N \) is the smallest positive integer such that \( S > 1,000,000. \) Determine exactly the value of \( N. \)

Solution to 8 is on \textbf{Solutions (page 3 of 3)}.\]
Problems (8 points)

9. In isosceles triangles $ABC$ and $PQR$, $\angle A \equiv \angle P$, $AB = PQ = x$, and $BC = QR = 1$, but the two triangles are NOT similar. If $x > 1$, determine exactly the value of $x$.

10. If $a > 0$ and $(a,b)$ is a solution to the system

$$|x + 2y| = 6 \text{ and } |x - 4y| = 3,$$

determine exactly the greatest possible value for $a + b$.

11. $z^{2019} = 1$ has 2019 solutions in the complex plane. Calculate the product of all 2019 solutions. Express your answer in the form $r \text{ cis } \theta$, where $0 \leq \theta < 2\pi$ and $r > 0$.

12. Compute the number of ordered pairs of positive integers $(x,y)$ for which $3x + 5y = 2019$.

Challenges (8 points)

13. There is only one four-digit integer of the form $aab\!b$ that is a perfect square. Find it.

14. $\triangle ABC$ has a right angle at $C$. Squares $ABDE$, $CAFG$, and $BCHI$ are constructed as in Figure 14.1. If $DI = 6$ and $EF = 7$, determine exactly $HG$.

Figure 14.1
8. \[ S = 2(1 + 2 + 3 + \ldots + N) = 2 \left( \frac{N(N + 1)}{2} \right) = N(N + 1). \sqrt{1000000} = 1000. \] So try \( N = 1000 \):
\[ 1000(1001) = 1001000 > 1000000. \] But is this the smallest? Try \( N = 999 \): \[ 999(1000) = 999000 < 1000000. \]

9. There are only two possible triangles but each can be labeled in several ways. One such labeling is shown below.
\[ \cos A = \frac{x^2 + x^2 - 1^2}{2 \cdot x \cdot x} = \frac{2x^2 - 1}{2x^2}. \] \[ \cos P = \frac{1^2 + x^2 - 1^2}{2 \cdot 1 \cdot x} = \frac{x}{2}. \] Therefore, \[ \frac{2x^2 - 1}{2x^2} = \frac{x}{2} \Rightarrow 4x^2 - 2 = 2x^3 \Rightarrow \\
2x^3 - 4x^2 + 2 = 0 \Rightarrow x^3 - 2x^2 + 1 = 0. \] Using synthetic division, the zeros are 1 and \( \frac{1 \pm \sqrt{5}}{2} \). Since \( x > 0 \) and \( x \neq 1 \), \( x = \frac{1 + \sqrt{5}}{2} \).

10. Let \( (a, b) \) be in \( Q I \). Then \( a + 2b = 6 \) and \( a - 4b = 3 \). Solving the system yields \( a = 5 \), \( b = .5 \), so \( a + b = 5.5 \). But what if \( (a, b) \) is in \( Q IV \)? Then \( a + 2b = \pm 6 \) and \( a + 4b = 3 \). Solving this system when using \( +6 \) yields \( a = 9 \), \( b = -1.5 \), so \( a + b = 7.5 \), and when using \( -6 \) yields \( a = -15 \), \( b = 4.5 \), not in \( Q IV \). So maximum sum is 7.5.

11. 
\[ 2019 - 1 = 0. \] By the Root-Coefficient Theorem, the product of the roots is \( \pm \) constant term/leading coefficient. It is (+) when \( n \) is even and (-) when \( n \) is odd. Therefore, the product is \(-1) = 1 = 1 \text{ cis } 0.

12. Slope is -3/5 and x-intercept is 673. Up 3, left 5 produces (668, 3). Since 673/5 = 134.6, one can repeat the process "up 3, left 5", 134 times.

13. Let \( N = aabb = 1000a + 100a + 10b + b = 1100a + 11b = 11(100a + b) \). Therefore, 11 must be a factor of \( 100a + b = a0b \). The divisibility test for 11 states that \( a - 0 + b = a + b \) is a multiple of 11. So there are only eight possibilities: 209 = 11·19, 308 = 11·28, 407 = 11·37, 506 = 11·46, 605 = 11·55, 704 = 11·64, 803 = 11·73, 902 = 11·82. Only one of these, 704, has a square as the second factor. So \( N = 7744 = 88^2 \).

14. See Figure 14.1. Rotate \( \triangle DBI \) 90° counter-clockwise about point \( B \), creating \( \triangle AI' C \) with \( AI' = 6 \). \( (2a)^2 + b^2 = 6^2 \).

Rotate \( \triangle AFE \) 90° clockwise about point \( A \), creating \( \triangle AF'B \) with \( F'B = 7 \). \( (2b)^2 + a^2 = 7^2 \). Then \( 4a^2 + b^2 = 36 \)
Therefore, \( 5a^2 + 5b^2 = 85 \Rightarrow a^2 + b^2 = 17 \Rightarrow c = x = \sqrt{17} \). (Can also be solved using Law of Cosines twice.)