2019-20 Meet 1, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

*NO CALCULATORS are allowed on this event.*

1. Express $\frac{4+5}{3 \cdot 4 + 4 + 5}$ as a quotient of two relatively prime integers.

2. Let $b$ be a positive integer. For how many values of $b$ is $21_b$ a two-digit number in base 10?

3. Determine exactly the smallest positive rational number which when divided by $\frac{4}{11}$ or $\frac{3}{22}$ or $\frac{5}{33}$ always yields an integer?

4. Determine the number of ordered triples of digits $(A, B, C)$, such that $\frac{\overline{AB}}{\overline{CA}} = 2$, that is, a decimal with a two-digit repetend divided by a decimal with a two-digit repetend equals 2.

Name: ___________________________ Team: ___________________________
1. Express \( \frac{4 + 5}{3 + 4} \) as a quotient of two relatively prime integers.

\[
\frac{4 + 5}{3 + 4} = \frac{16 + 15}{12} = \frac{20}{12} = \frac{5}{3}
\]

2. Let \( b \) be a positive integer. For how many values of \( b \) is \( 21_b \), a two-digit number in base 10?

\[
10 \leq 2b + 1 \leq 99 \rightarrow 4.5 \leq b \leq 49, \text{and so } b \text{ takes on values from 5 to 49, inclusive. There are 49 – 5 + 1 = 45 values.}
\]

3. Determine exactly the smallest positive rational number which when divided by \( \frac{4}{11} \) or \( \frac{3}{22} \) or \( \frac{5}{33} \) always yields an integer?

The numerator must be the LCM of 4, 3, and 5 while the denominator must be the GCD of 11, 22, 33. Thus the number is \( \frac{60}{11} \). (\( \frac{60}{11} + \frac{4}{11} = \frac{15}{11} = 15, \frac{60}{22} + \frac{3}{22} = \frac{20}{11} = 40, \text{and } \frac{60}{33} + \frac{5}{33} = \frac{36}{11} \).)

4. Determine the number of ordered triples of digits \( (A, B, C) \), such that \( \overline{AB} + \overline{CA} = 2 \), that is, a decimal with a two digit repand divided by a decimal with a two digit repand equals 2.

Note that \( \frac{M + N}{99} \). Therefore, \( \frac{10A + B}{99} = 2 \rightarrow 10A + B = 20C + 2A \rightarrow 8A + B = 20C \).

If \( C = 1 \) then \( A = 2 \) and \( B = 4 \). If \( C = 2 \), \( A = 5 \) and \( B = 0 \) or \( A = 4 \) and \( B = 8 \). If \( C = 3 \), \( A = 7 \) and \( B = 4 \). If \( C = 4 \), \( A = 9 \) and \( B = 8 \). If \( C > 5 \), then either \( A \geq 10 \) or \( B \geq 10 \). So there are 5 triples satisfying the problem.
1. In Figure 1, $ABCDEFGH$ is a cube. What is $m\angle EBD$?
(Hint: The answer is not $90^\circ$.)

2. In Figure 2, determine exactly the sum of the angles labelled 1 through 10.

3. The interior angles of a convex polygon increase in the following linear progression: $100^\circ$, $108^\circ$, $116^\circ$, ... . Determine the number of sides of the polygon.

4. The sides of right triangle $ABC$ are $a$, $a+7d$, and $a+9d$ with $a$ and $d$ being integers. What is the smallest possible perimeter of $\triangle ABC$?
1. In Figure 1, ABCDEFGH is a cube. What is \( m \angle EBD \)?
   (Hint: The answer is not 90°.)

Draw \( ED \). Since each segment is the diagonal of a face of the cube, \( \triangle EBD \) must be equilateral. Therefore, \( m \angle EBD = 60° \).

2. In Figure 2, determine exactly the sum of the angles labelled 1 through 10.

\[
5(180°) - 3(180°) = 2(180°) = 360°
\]

3. The interior angles of a convex polygon increase in the following linear progression: 100°, 108°, 116°, … . Determine the number of sides of the polygon.

The exterior angles of this polygon must also form a linear progression: 80°, 72°, 64°, … . Since the exterior angles of any convex polygon add up to 360°, continue adding to the progression until the total reaches 360°. 80° + 72° + 64° + 56° + 48° + 40° = 360°. Therefore this must be a 6-sided polygon.

4. The sides of right triangle \( \triangle ABC \) are \( a \), \( a + 7d \), and \( a + 9d \) with \( a \) and \( d \) being integers. What is the smallest possible perimeter of \( \triangle ABC \)?

\[
a^2 + (a + 7d)^2 = (a + 9d)^2 \rightarrow a^2 + a^2 + 14ad + 49d^2 = a^2 + 18ad + 81d^2 \rightarrow a^2 - 4ad - 32d^2 = 0.
\]

This factors into \( (a + 4d)(a - 8d) = 0 \). So \( a = -4d \) or \( a = 8d \). The first solution is impossible but the second yields a triangle with sides 8d, 15d, and 17d. If \( d = 1 \), the perimeter would be 40.
1. Determine exactly the value of \( \sin \frac{\pi}{3} + \tan \frac{\pi}{4} + \cos \frac{\pi}{6} \).

2. Determine exactly the smallest positive integer \( n \) such that \( \sec(400^\circ) \cdot \sin(n^\circ) = 1 \).

3. \( \triangle ABC \) has a right angle at \( B \). If \( BC = 1 \) and \( \cos A = \frac{1}{3} \), determine exactly the perimeter of the triangle.

4. In trapezoid \( ABCD \), \( AB \parallel CD \). If \( AB = 6 \), \( BC = 8 \), \( CD = 15 \), and \( AD = 4 \), determine exactly the value of \( \cos A + 2 \cos B \).
1. Determine exactly the value of \( \sin \frac{\pi}{3} + \tan \frac{\pi}{4} + \cos \frac{\pi}{6} \).

\[
\frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2} = 1 + \sqrt{3}
\]

2. Determine exactly the smallest positive integer \( n \), such that \( \sec(400^\circ) \cdot \sin(n^\circ) = 1 \).

\[
\sec(400^\circ) = \frac{1}{\cos(400^\circ)} = \frac{1}{\cos(40^\circ)}. \text{ Therefore, } \sin(n^\circ) = \cos(40^\circ). \text{ But } \cos(40^\circ) = \sin(90^\circ - 40^\circ) = \sin(50^\circ).
\]

3. \( \triangle ABC \) has a right angle at \( B \). If \( BC = 1 \) and \( \cos A = \frac{1}{3} \), determine exactly the perimeter of the triangle.

\[
\text{In Figure 3.1, } \cos A = \frac{AB}{AC} = \frac{x}{3x} \text{ for some positive } x. \text{ Therefore, } 1^2 + x^2 = 9x^2 \text{ and } x^2 = \frac{1}{8} \text{ and } x = \frac{1}{\sqrt{8}} = \frac{\sqrt{2}}{4}. \text{ The perimeter is } x + 3x + 1 = 4x + 1 = \sqrt{2} + 1.
\]

4. In trapezoid \( ABCD \), \( AB \parallel CD \). If \( AB = 6 \), \( BC = 8 \), \( CD = 15 \), and \( AD = 4 \), determine exactly the value of \( \cos A + 2 \cos B \).

\[
\text{In Figure 4.1, draw altitudes } \overline{AS} \text{ and } \overline{BT}. \text{ Then } \cos D = \frac{DS}{4} \text{ and } \cos C = \frac{CT}{8}. \text{ Therefore, } \cos D + 2 \cos C = \frac{9}{4}. \text{ Since } \overline{AB} \parallel \overline{CD}, \cos A = -\cos D \text{ and } \cos B = -\cos C. \text{ So } \cos A + 2 \cos B = -\frac{9}{4}.
\]
Minnesota State High School Mathematics League
2019-20 Meet 1, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

1. Given \( f(x) = 3x^5 + 5x^3 - 2x^2 + 82 \), determine exactly \( f^{-1}(f(1)) \).

2. \( f(x) = x^2 + bx + 12 \). Determine for how many integer values of \( b \), \( f(x) \) has non-real zeros.

3. \( f(x) = ax^2 \) with \( a > 0 \). An equilateral triangle with side length \( k \) is placed on the parabola so that one of its vertices is on the vertex of the parabola and the other two vertices are on \( f(x) \). Write a formula for \( a \), the leading coefficient of \( f(x) \), in terms of \( k \). (Be sure to simplify.)

4. \( f(x) = -(x-r)(x-t) \) with \( t > r \). A right triangle is placed on \( f(x) \) such that two of its vertices are \((r,0)\) and \((t,0)\) and its right angle vertex is on \( f(x) \). Write a formula for the area of this triangle in terms of \( r \) and \( t \).

Name: ___________________________  Team: ___________________________
1. Given \( f(x) = 3x^5 + 5x^3 - 2x^2 + 82 \), determine exactly \( f\left(f^{-1}(f(1))\right) \).

\[
f^{-1}(f(x)) = x. \text{ So } f\left(f^{-1}(f(1))\right) = f(1) = 3 + 5 - 2 + 82 = 88.
\]

2. \( f(x) = x^2 + bx + 12 \). Determine for how many integer values of \( b \), \( f(x) \) has non-real zeros.

To have non-real zeros the determinant must be negative. So \( b^2 - 4(1)(12) < 0 \Rightarrow b^2 < 48 \). Since \( b \) is an integer, \( -6 \leq b \leq 6 \).

3. \( f(x) = ax^2 \) with \( a > 0 \). An equilateral triangle with side length \( k \) is placed on the parabola so that one of its vertices is on the vertex of the parabola and the other two vertices are on \( f(x) \). Write a formula for \( a \), the leading coefficient of \( f(x) \), in terms of \( k \). (Be sure to simplify.)

\[
a = \frac{2\sqrt{3}}{k}.
\]

The point \( \left( \frac{k}{2}, \frac{k\sqrt{3}}{2} \right) \) will be a point on this parabola. Therefore, \( \frac{k\sqrt{3}}{2} = a\left(\frac{k}{2}\right)^2 \Rightarrow a = \frac{4}{k^2} \cdot \frac{k\sqrt{3}}{2} = \frac{2\sqrt{3}}{k} \).

4. \( f(x) = -(x-r)(x-t) \) with \( t > r \). A right triangle is placed on \( f(x) \) such that two of its vertices are \( (r,0) \) and \( (t,0) \) and its right angle vertex is on \( f(x) \). Write a formula for the area of this triangle in terms of \( r \) and \( t \).

The third vertex must be \( (n, f(n)) \) for some value of \( n \). Since the sides to the other two vertices are perpendicular at this point, \( \frac{f(n) - 0}{n - r} = -\frac{n - t}{f(n) - 0} \Rightarrow (f(n))^2 = -(n-r)(n-t) \Rightarrow f(n) = 1 \). Therefore, the altitude of the triangle is 1 and the area is \( \frac{1}{2}(t-r)(1) \).
If the two-digit number $AB = k + 6$ and the two-digit number $BA = 3k - 20$, determine exactly the integer value of $BA - AB$.

Let $a$ and $b$ be positive integers. If $ab < 10$, determine the largest possible integer value of the infinite continued fraction: $a + \frac{3}{b + \frac{3}{a + \frac{3}{b + \frac{3}{a + \cdots}}}}$.

Let $a$ and $b$ be three-digit positive integers with $a < b$. Compute the number of ordered pairs $(a, b)$ such that $a + b$ is a four-digit number.

In $\triangle ABC$, point $E$ is on $AC$ so that $AB = AE$. If $m\angle ABC - m\angle ACB = 20^\circ$, determine exactly $m\angle EBC$.

In pentagon $ABCDE$, $AB = BC = DE = EA = \frac{1}{2}$, $\angle A = \angle C = \angle D = 128^\circ$, and $\angle B = \angle E$. If $CD = \sin x^\circ - \sin y^\circ$, where $x$ and $y$ are integers between 0 and 90, determine exactly the value of $x + y$.

Let $p$ be a positive prime and $x^2 + p^2x + 184p = 4$. Determine the smallest value of $p$ so that the quadratic equation has two distinct integer roots.
1. If the two-digit number $AB = k + 6$ and the two-digit number $BA = 3k - 20$, determine exactly the integer value of $BA - AB$.

2. Let $a$ and $b$ be positive integers. If $ab < 10$, determine the largest possible integer value of the infinite continued fraction: $a + \cfrac{3}{b + \cfrac{3}{a + \cfrac{3}{b + \cfrac{3}{a + ...}}}}$.

3. Let $a$ and $b$ be three-digit positive integers with $a < b$. Compute the number of ordered pairs $(a,b)$ such that $a + b$ is a four-digit number.

4. In $\triangle ABC$, point $E$ is on $AC$ so that $AB = AE$. If $m\angle ABC - m\angle ACB = 20^\circ$, determine exactly $m\angle EBC$.

5. In pentagon $ABCDE$, $AB = BC = DE = EA = \frac{1}{2}$, $\angle A = \angle C = \angle D = 128^\circ$, and $\angle B = \angle E$. If $CD = \sin x^\circ - \sin y^\circ$, where $x$ and $y$ are integers between 0 and 90, determine exactly the value of $x + y$.

6. Let $p$ be a positive prime and $x^2 + p^2x + 184p = 4$. Determine the smallest value of $p$ so that the quadratic equation has two distinct integer roots.
1. \((10A + B) - (10B + A) = (k + 6) - (3k - 20) \rightarrow 9(A - B) = -2(k - 13).\) Therefore, \(91(k - 13).\) So \(k\) could be 13, 22, 31, 40, … If \(k = 13\), then \(AB = 19\) and \(BA = 19\), not 91. If \(k = 22\), then \(AB = 28\) and \(BA = 46\), not 82. If \(k = 31\), then \(AB = 37\) and \(BA = 73\) … works! Therefore, 73 – 37 = 36.

2. The continued fraction reduces to \(x = a + \frac{3}{b + \frac{3}{x}} = a + \frac{3x}{bx + 3} = \frac{abx + 3a + 3x}{bx + 3}.\) Therefore,

\[
bx^2 + 3x = abx + 3a + 3x \rightarrow bx^2 - abx - 3a = 0.\]

So \(x = \frac{ab \pm \sqrt{(ab)^2 - 4 \cdot b \cdot (-3a)}}{2b} = \frac{ab \pm \sqrt{ab(ab + 12)}}{2b}.

Note that if \(ab = 4\), then \(ab + 12 = 16\) and \(x = \frac{4 \pm 8}{2b} = \frac{6}{b}.\) The largest this can be is \(x = 6.\) (It can be shown that if \(ab\) is any integer > 4, there are no integer solutions.)

3. If \(a = 100\), then the range of possible values for \(b\) is \([900, 999]\) (100 values). If \(a = 101\), then the range of possible values for \(b\) is \([899, 999]\) (101 values). If \(a = 102\), then the range of possible values for \(b\) is \([898, 999]\) (102 solutions). This pattern continues for all values of \(a\) from 100 to 499 (There will be a possible values for \(b\)). But for \(a = 500\), the range of possible values for \(b\) is \([501, 999]\) (499 values). If \(a = 501\), the range of possible values for \(b\) is \([502, 999]\) (498 values). If \(a = 502\), the range of possible values for \(b\) is \([503 … 999]\) (497 values). This pattern continues for all values of \(a\) from 500 to 998. The range of possible values for \(b\) is \([a + 1, 999]\) (999 - \(a\) values). Therefore, there are \((100 + 101 + 102 + … + 499) + (499 + 498 + 497 + … + 1)\) possible pairs or

\[
\frac{100 + 499}{2} \cdot 400 + \frac{1 + 499}{2} \cdot 499 = (200)(599) + (250)(499) = 244550 \text{ pairs}.
\]

4. Let \(\angle EBC = x\) and \(\angle ACB = y\) (See Figure 4.1). It is given that \(\angle ABC = \angle ACB + 20 = y + 20.\) Because \(AB = AE, \angle AEB = \angle ABE.\) By the exterior angle theorem, \(\angle AEB = x + y.\) But \(\angle ABC = \angle ABE + \angle EBC.\) Therefore, \(y + 20 = x + y + x \rightarrow 20 = 2x \rightarrow x = 10.

5. Draw \(\overline{BE}\) and altitudes \(\overline{AY}, \overline{DX}\) and \(\overline{CZ}\) onto \(\overline{BE}\) and note \(CD = EB - 2 \cdot EX\) (See Figure 5.1). Since \(\angle B = \angle E, \overline{BCDE}\) is an isosceles trapezoid with \(\overline{BE} \parallel \overline{CD}.\) \(\angle EAY = \angle BAY = \frac{1}{2} \cdot 128^\circ = 64^\circ,\) so \(EY = BY = \frac{1}{2} \cdot \sin 64^\circ.\) Thus \(EB = \sin 64^\circ.\) \(\angle BCZ = \angle EDX = 128^\circ - 90^\circ = 38^\circ,\) so \(EX = BZ = \frac{1}{2} \cdot \sin 38^\circ.\) Therefore, \(CD = XZ = EB - 2 \cdot EX = \sin 64^\circ - \sin 38^\circ\) and \(64 + 38 = 102.

6. The discriminant will be \(p^4 - 4(184p - 4) = p^4 - 736p + 16.\) For there to be two distinct integer solutions \(p^4 - 736p + 16\) must be a positive perfect square. Therefore, \(p^3 > 736\) or \(p > 9.\) The next largest prime is 11 and evaluating the discriminant with \(p = 11\) gives 6561 = 81^2, giving roots –20 and –101.