2018-19 Meet 3, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 12 minutes for this event.

1. Determine exactly the coordinates of the intersection of the lines
\[ \frac{x}{5} + \frac{y}{2} = 1 \text{ and } \frac{-3x}{4} + \frac{y}{2} = 1. \]

2. Determine exactly the ordered triple \((x, y, z)\) that satisfies this system of equations:
\[
\begin{align*}
7x + 2y - 4z &= 19 \\
5x + 3y - 3z &= 15 \\
5x - 3y + 3z &= 15
\end{align*}
\]

3. Apples and melons are on sale at the local farmers’ market. Elaine buys 10 apples and 5 melons, pays with $10.00 and receives change. Xi buys 5 apples and 10 melons, pays with $10.00 and also receives change. Elaine and Xi give me their change and I add 10 cents and buy 3 apples and 1 melon, receiving 2 cents in change. Jorge buys 20 apples and 20 melons with $25.00 and receives $1.00 in change. How much does each apple cost?

4. A pedestrian and a cyclist leave point A for point B simultaneously at 10:00. Reaching point B, the cyclist immediately turns around and, while returning to point A, passes the pedestrian at 10:20. When the cyclist reaches point A again, she immediately turns around and catches up to the pedestrian at 10:30. If the cyclist and the pedestrian each traveled at a uniform rate, what time will it be when the pedestrian finally reaches point B?
1. Determine exactly the coordinates of the intersection of the lines

\[
\frac{x}{5} + \frac{y}{2} = 1 \quad \text{and} \quad \frac{-3x}{4} + \frac{y}{2} = 1.
\]

Notice: \( \frac{x}{5} = \frac{-3x}{4} \Rightarrow x = 0. \) Then \( \frac{y}{2} = 1 \Rightarrow y = 2. \)

2. Determine exactly the ordered triple \((x, y, z)\) that satisfies this system of equations:

\[
\begin{align*}
7x + 2y - 4z &= 19 \quad \text{Adding the last two equations gives} \\
5x + 3y - 3z &= 15 \quad 10x = 30 \Rightarrow x = 3. \ \text{Substituting } x = 3 \text{ into the} \\
5x - 3y + 3z &= 15 \quad \text{second equation gives} \ 3y - 3z = 0 \Rightarrow y = z. \\
\end{align*}
\]

Substituting into the first equation gives

\[
21 + 2y - 4y = 0 \Rightarrow -2y = -2 \Rightarrow y = 1. \ \text{So } z = 1.
\]

3. Apples and melons are on sale at the local farmers’ market. Elaine buys 10 apples and 5 melons, pays with $10.00 and receives change. Xi buys 5 apples and 10 melons, pays with $10.00 and also receives change. Elaine and Xi give me their change and I add 10 cents and buy 3 apples and 1 melon, receiving 2 cents in change. Jorge buys 20 apples and 20 melons with $25.00 and receives $1.00 in change. How much does each apple cost?

The equations are: \(10A + 5M + x = 10, \ 5A + 10M + y = 10, \ 3A + 1M = x + y + .08, \) and \(20A + 20M = 24. \) Adding the first two gives: \(15A + 15M + x + y = 20. \) Solving the third for \(x + y\) and substituting gives: \(18A + 16M = 20.08\) and solving by elimination with the last equation gives \(A = .44\).

4. A pedestrian and a cyclist leave point A for point B simultaneously at 10:00. Reaching point B, the cyclist immediately turns around and, while returning to point A, passes the pedestrian at 10:20. When the cyclist reaches point A again, she immediately turns around and catches up to the pedestrian at 10:30. If the cyclist and the pedestrian each traveled at a uniform rate, what time will it be when the pedestrian finally reaches point B?

Let \(x = \) the distance from A to B and \(y = \) the distance the pedestrian travelled in 20 minutes. If the rate of the cyclist is \(C\) and the rate of the pedestrian is \(P,\) then

\[
x + (x - y) = 2x - y = (C)(20) \quad \text{and} \quad y = (P)(20).
\]

In the next 10 minutes, the pedestrian travels a distance of \(\frac{y}{2}. \) So

\[
2y + \frac{y}{2} = (C)(10) \quad \text{and} \quad \frac{y}{2} = (P)(10).
\]

Combining the last three equations gives \(C = 5P. \) Substituting this into the first equation gives

\[
2x - 20P = 5P(20) \Rightarrow 2x = 120P \Rightarrow \frac{x}{P} = 60. \ \text{Therefore, it will be 11:00.}\]
Minnesota State High School Mathematics League

2018-19 Meet 3, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

1. If the areas of an equilateral triangle and a square are equal, determine exactly the ratio of the side of the square to the side of the triangle. Express your answer in the form \((\frac{a}{b})^c\).

2. Given a \(3 \times 3 \times 3\) cube, a \(1 \times 1 \times 1\) cube is cut out of the middle of each face. What is the surface area of the resulting solid?

\[\text{MN} = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_] \]

\[\text{If the areas of an equilateral triangle and a square are equal, determine exactly the ratio of the side of the square to the side of the triangle. Express your answer in the form } \left(\frac{a}{b}\right)^c.\]

3. \(ABCD\) is a square whose side length is 32. \(DMBN\) is a rhombus whose vertices lie on the diagonals of the square. If the area of the rhombus is 75% of the area of the square, determine exactly the length of \(MN\).

4. The solid shown in Figure 4 has a rectangular base \(BEFC\) with equilateral triangles \(ABC\) and \(DEF\) tilting toward each other. If \(M\) and \(N\) are the midpoints of \(BC\) and \(EF\), then \(\angle AMN = \angle DNM = 60^\circ\). If \(BC = 4\sqrt{3}\) and \(CF = 28\), determine exactly the volume of the solid.

\[\text{Figure 3}\]

\[\text{Figure 4}\]
SOLUTIONS

1. If the areas of an equilateral triangle and a square are equal, determine exactly the ratio of the side of the square to the side of the triangle. Express your answer in the form \( \left( \frac{a}{b} \right)^c \).

Let the side of the triangle have length \( s \) and the side of the square have length \( x \). Then

\[
\frac{s^2 \sqrt{3}}{4} = x^2 \Rightarrow x^2 = \frac{\sqrt{3}}{4} s^2 \Rightarrow x = \frac{\sqrt{3}}{2} s = \frac{3^{\frac{3}{4}}}{2} = \left( \frac{3}{16} \right)^{\frac{3}{4}}.
\]

2. Given a \( 3 \times 3 \times 3 \) cube, a \( 1 \times 1 \times 1 \) cube is cut out of the middle of each face. What is the surface area of the resulting solid?

The \( 1 \times 1 \) square on the face of the cube is dropped 1 unit into the cube and the four sides of the cut out hole add four square units to each face of the cube. So the resulting surface area is \( 6 \cdot 9 + 6 \cdot 4 = 78 \) square units.

3. \( ABCD \) is a square whose side length is 32. \( DMBN \) is a rhombus whose vertices lie on the diagonals of the square. If the area of the rhombus is 75\% of the area of the square, determine exactly the length of \( MN \).

\[
\frac{1}{2} (MN)(DB) = .75(32)^2 \Rightarrow (MN)(32\sqrt{2}) = 1536 \Rightarrow MN = \frac{48}{\sqrt{2}} = 24\sqrt{2}.
\]

4. The solid shown in Figure 4 has a rectangular base \( BEFC \) with equilateral triangles \( ABC \) and \( DEF \) tilting toward each other. If \( M \) and \( N \) are the midpoints of \( BC \) and \( EF \), then \( \angle AMN = \angle DNM = 60^{\circ} \). If \( BC = 4\sqrt{3} \) and \( CF = 28 \), determine exactly the volume of the solid.

Drop an altitude \( \overline{AT} \perp BEFC \). \( \overline{AM} \), the altitude of \( \triangle ABC \), has a length of 6.

\( AMT \) is a 30-60-90 triangle, so \( MT = 3 \) and \( AT = 3\sqrt{3} \). Similarly, drop an altitude \( \overline{DS} \perp BEFC \). \( NS = 3 \) and \( DS = 3\sqrt{3} \). Using vertical cuts through \( A \) and \( D \), slice the solid into a prism \( AWXYZD \) and two congruent pyramids with rectangular bases. The volume of the solid then is

\[
\frac{(4\sqrt{3}) \cdot (3\sqrt{3})}{2} \cdot 22 + 2 \left( \frac{1}{3} \right) \left( 4\sqrt{3} \right) (3) \cdot 3\sqrt{3} = 468.
\]
1. \( z = 1 + 6i \). \( w = z \cdot \bar{z} \), where \( \bar{z} \) is the conjugate of \( z \). Determine exactly the value of \( w \).

2. If \( z = \text{cis}(30^\circ) \), determine exactly the value of \( z^3 + \frac{1}{\bar{z}^3} \).

\( (z = r\text{cis}(\theta) \) is shorthand notation for the complex number \( z = r \cos \theta + r \sin \theta \).

3. If \( \cos(\arctan(x)) = x \), then \( x^2 \) can be expressed exactly in the form \( \frac{a + \sqrt{b}}{2} \).

Calculate \( a + b \).

4. Determine exactly, in terms of radians, the value of \( \arctan(2 - \sqrt{3}) + \arctan(2 + \sqrt{3}) + \arctan(\sqrt{3}) \).
1. \( z = 1 + 6i \). \( w = z \cdot \overline{z} \), where \( \overline{z} \) is the conjugate of \( z \). Determine exactly the value of \( w \).

\[
w = (1 + 6i)(1 - 6i) = 1 - 6i + 6i - 36i^2 = 37.
\]

2. If \( z = cis(30^\circ) \), determine exactly the value of \( z^3 + \frac{1}{z^3} \).

\[
( z = rcis(\theta) \) is shorthand notation for the complex number \( z = r \cos \theta + r \sin \theta i \).
\]

\[
z^3 + \overline{z^3} = cis(90^\circ) + cis(-90^\circ) = \cos 90^\circ + i \sin 90^\circ + \cos(-90^\circ) + i \sin(-90^\circ) = 0 + i0 - i0 = 0.
\]

3. If \( \cos(\arctan(x)) = x \), then \( x^2 \) can be expressed exactly in the form \( \frac{a + \sqrt{b}}{2} \). Calculate \( a + b \).

\[
\text{Let } A = \arctan x. \text{ Then } \cos A = x \text{ and } \tan A = x. \text{ Consider right triangle } ABC, \text{ with hypotenuse of length 1. Since } \cos A = x, \ AC = x. \text{ Since } \tan A = x, \ BC = x^2.
\]

Therefore, \( x^2 + (x^2)^2 = 1 \implies y + y^2 = 1 \implies y^2 + y - 1 = 0 \). So \( y = x^2 = \frac{-1 + \sqrt{5}}{2} \)

and \( a + b = -1 + 5 = 4 \).

4. Determine exactly, in terms of radians, the value of \( \arctan(2 - \sqrt{3}) + \arctan(2 + \sqrt{3}) + \arctan(\sqrt{3}) \).

\[
\text{Let } A = \arctan(2 - \sqrt{3}) \text{ and } B = \arctan(2 + \sqrt{3}). \text{ Then } \tan A = 2 - \sqrt{3} \text{ and } \tan B = 2 + \sqrt{3}. \text{ Let } x = A + B.
\]

Then \( \tan x = \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{2 - \sqrt{3} + 2 + \sqrt{3}}{1 - (2 - \sqrt{3})(2 + \sqrt{3})} = \frac{4}{1 - (4 - 3)} = \frac{4}{0} \). Therefore, \( x = \frac{\pi}{2} \). Let \( C = \arctan(\sqrt{3}) \). Then \( \tan C = \sqrt{3} \implies C = \frac{\pi}{3} \). So the value of the expression is \( \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6} \).
1. Determine exactly the value of \( \left( \frac{1}{64} \right)^{-\frac{1}{2}} + \left( \frac{1}{64} \right)^{-\frac{1}{3}} + \left( \frac{1}{64} \right)^{-\frac{1}{4}} \).

2. Determine exactly the value of \( \log_{12} 24 \cdot \log_{12} 72 \).

3. Let \( Q = \log_{3} 15 \). If the number \( \log_{3} 375 \) can be determined exactly in the form \( a \cdot Q + b \), for some integers \( a \) and \( b \), determine \( a \) and \( b \).

4. Let \( b = 3^{25} \). Determine exactly the value of \( x \), \( x \neq 1 \), given that \( \sqrt[\log_b x]{} = \log_{\sqrt[\log_b x]{} b} x + \log_b \sqrt{x} \).
1. Determine exactly the value of \( \left( \frac{1}{64} \right)^{-\frac{1}{4}} + \left( \frac{1}{64} \right)^{\frac{1}{4}} + \left( \frac{1}{64} \right)^{-\frac{1}{4}} \).

\[
64 + \sqrt[4]{64} + \sqrt[4]{64} = 64 + 8 + 4 + 2 = 78.
\]

2. Determine exactly the value of \( \log_{12} 24 + \log_{12} 72. \)

This is equal to \( \log_{12} (24 \cdot 72) = \log_{12} ((12 \cdot 2)(12 \cdot 6)) = \log_{12} (12^2 \cdot 2 \cdot 6) = \log_{12} (12^3) = 3. \)

3. Let \( Q = \log_3 15. \) If the number \( \log_3 375 \) can be determined exactly in the form \( a \cdot Q + b, \) for some integers \( a \) and \( b, \) determine \( a \) and \( b. \)

\[
Q = \log_3 3 + \log_3 5 = 1 + \log_3 5. \text{ So } \log_3 5 = Q - 1. \text{ But } \log_3 375 = \log_3 3 + \log_3 125 = 1 + 3(Q - 1) = 3Q - 2. \text{ Therefore, } a = 3 \text{ and } b = -2.
\]

4. Let \( b = 3^{25}. \) Determine exactly the value of \( x, \ x \neq 1, \) given that

\[
\sqrt{\log_b x} = \log_b x + \log_b \sqrt{x}.
\]

\[
\sqrt{\log_b x} = \frac{\log_b x}{\log_b \sqrt{b}} + \log_b \left( x^{\frac{1}{2}} \right). \text{ Let } m = \log_b x. \text{ Then } \sqrt{m} = \frac{m}{2} + \frac{1}{2}m \Rightarrow \sqrt{m} = \frac{5m}{2} \Rightarrow \frac{25}{4}m^2 \Rightarrow 0 = \frac{25}{4}m^2 - m \Rightarrow 0 = m \left( \frac{25}{4}m - 1 \right) \Rightarrow m = 0 \text{ or } m = \frac{4}{25}. \text{ } m = 0 \text{ is extraneous because it causes } x \text{ to equal 1. Therefore, } \log_b x = \frac{4}{25}, \ x = b^{\frac{4}{25}} \Rightarrow x = \left( 3^{25} \right)^{\frac{4}{25}} = 3^4 = 81.\]
1. The intersections of the lines $3x - 2y = -5$, $x + 4y = 17$, and $9x + 8y = 125$ determine the vertices of a triangle. Determine exactly the area of this triangle.

2. $ABCDEFG$ is a regular heptagon with side length of 1. Let length of $AD$ equal $x$. Write an expression for the length of $GE$ in terms of $x$.

3. In $\triangle ABC$, if $\cos A = 2018 \cos B \cos C$ and $\sin A = 2018 \sin B \sin C$, determine exactly the value of $\tan A$.

4. In a regular pentagon $ABCDE$, the ratio of the area of $\triangle ACE$ to the area of $\triangle CDE$ can be written as $k \sin \theta$. Compute the ordered pair $(k, \theta)$, where acute angle $\theta$ is in degrees.

5. Let $A = \left( \frac{1}{1} \right) \cdot \left( \frac{1}{2} \right) \cdot \left( \frac{1}{3} \right) \cdot \ldots \cdot \left( \frac{1}{8} \right)$. Compute the least positive integer $n$ such that $A^n$ is an integer.

6. Given that $\log_2 \left( \log_2 (x) \right) = \log_{16} \left( \log_{16} (x) \right)$ for some $x > 1$, determine exactly the value of $\log_4 \left( \log_4 (x) \right)$. Express answer as a quotient of two relatively prime integers.

Team: ____________________________
1. The intersections of the lines $3x - 2y = -5$, $x + 4y = 17$, and $9x + 8y = 125$ determine the vertices of a triangle. Determine exactly the area of this triangle.

2. $ABCDEFG$ is a regular heptagon with side length of 1. Let length of $AD$ equal $x$. Write an expression for the length of $GE$ in terms of $x$.

Graders: To check any alternate answers, on your calculators, store "1 + 2 cos(360 / 7)" into $x$ and press enter. Then type in the student's expression and press enter. If you get 1.8019377, their answer is correct.

3. In $\triangle ABC$, if $\cos A = 2018 \cos B \cos C$ and $\sin A = 2018 \sin B \sin C$, determine exactly the value of $\tan A$.

4. In a regular pentagon $ABCDE$, the ratio of the area of $\triangle ACE$ to the area of $\triangle CDE$ can be written as $k \sin \theta$. Compute the ordered pair $(k, \theta)$, where acute angle $\theta$ is in degrees.

5. Let $A = \left(1^1\right) \cdot \left(2^2\right) \cdot \left(3^3\right) \cdots \left(8^8\right)$. Compute the least positive integer $n$ such that $A^n$ is an integer.

6. Given that $\log_2 (\log_2 (x)) = \log_{16} (\log_{16} (x))$ for some $x > 1$, determine exactly the value of $\log_4 (\log_4 (x))$. Express answer as a quotient of two relatively prime integers.
1. The three intersections of the three pairs of lines are (1,4), (5,10), (13,1). Let \[ A = \begin{bmatrix} 1 & 4 & 1 \\ 5 & 10 & 1 \\ 13 & 1 & 1 \end{bmatrix} \] and use determinants to find the area: \( \text{abs} \left( \frac{1}{2} \cdot \text{det}([A]) \right) = 42. \)

2. Because of the symmetry of the regular heptagon, \( AD = AE = GD = x \). All isosceles trapezoids are cyclic, so use Ptolemy’s Theorem: \( AG \cdot DE + GE \cdot AD = AE \cdot GD \Rightarrow 1 \cdot 1 + GE \cdot x = x \cdot x \Rightarrow GE = \frac{x^2 - 1}{x}. \)

3. Subtract the two equations:
\[
\cos A - \sin A = 2018(\cos B \cos C - \sin B \sin C)
= 2018 \cos(B + C)
= 2018 \cos(\pi - A)
= -2018 \cos A.
\]
Therefore, \( 2019 \cos A = \sin A \Rightarrow 2019 = \frac{\sin A}{\cos A} \Rightarrow \tan A = 2019. \)

4. See labels in Figure 4.1. \[
\begin{bmatrix} \text{ACE} \\ \text{CDE} \end{bmatrix} = \frac{1}{4} x^2 \sin 36^\circ = \frac{1}{2} x^2 \sin 36^\circ = \frac{x^2 \sin 36^\circ}{2 \sin 36^\circ \cos 36^\circ} = \frac{x^2}{2 \cos 36^\circ}. \]
By the Law of Cosines,
\[
x^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot \cos 108^\circ = 2 + 2 \cos 72^\circ = 2 + 2(2 \cos^2 36^\circ - 1) = 2 + 4 \cos^2 36^\circ - 2 = 4 \cos^2 36^\circ. \]
Therefore,
\[
\frac{\text{ACE}}{\text{CDE}} = \frac{4 \cos^2 36^\circ}{2 \cos 36^\circ} = 2 \cos 36^\circ = 2 \sin 54^\circ.
\]

5. \( A = 2^\frac{1}{2} \cdot 3^\frac{1}{4} \cdot (2^\frac{1}{3} \cdot 5^\frac{1}{5} \cdot 7^\frac{1}{7})^\frac{1}{3} = 2^{\frac{1}{2} + \frac{1}{3} + \frac{1}{7}} \cdot 3^{\frac{1}{4}} \cdot 5^{\frac{1}{5}} \cdot 7^{\frac{1}{7}} = 2^{\frac{11}{14}} \cdot 3^{\frac{1}{4}} \cdot 5^{\frac{1}{5}} \cdot 7^{\frac{1}{7}}. \)
Therefore, 
\( n = \text{lcm}(24, 2, 5, 7) = 840. \)

6. Let \( y = \log_4 (\log_4 x) \). Using Change of Base Law twice:
\[
\log_2 (\log_2 x) = \frac{\log_4 (\log_4 x)}{\log_4 2} = \frac{\log_4 (\log_4 x)}{\frac{1}{2}} = 2 \log_4 (2 \log_4 x) = 2(\log_4 2 + \log_4 (\log_4 x)) = 2\left(\frac{1}{2} + y\right) = 1 + 2y.
\]
\[
\log_{16} (\log_{16} x) = \frac{\log_4 (\log_{16} x)}{\log_4 16} = \frac{\log_4 (\log_{16} x)}{2} = \frac{1}{2} \log_4 \left(\frac{1}{2} \log_4 x\right) = \frac{1}{2} \left(\log_4 \frac{1}{2} + \log_4 (\log_4 x)\right) = \frac{1}{2} \left(-\frac{1}{2} + y\right) = -\frac{1}{4} + \frac{1}{2} y.
\]
Therefore, \( 1 + 2y = -\frac{1}{4} + \frac{1}{2} y \Rightarrow 4 + 8y = -1 + 2y \Rightarrow y = -\frac{5}{6}. \)