1. If \( p^2 = 2020 + q^2 \) and \( p = 10 + q \), compute \( p + q \).

2. If \( a \) and \( b \) are positive real numbers and \( \frac{a^2 + b^2}{1 + \frac{1}{a^2 + b^2}} = 10 \), determine exactly the value of
\[
\frac{a^3 + b^3}{1 + \frac{1}{a^3 + b^3}}
\]

3. Determine exactly the value for \( x \), given that \( x + 2\sqrt{x} = 2 \).

4. Let \( P = \frac{1}{2} + \frac{3}{4} + \frac{5}{6} + \ldots + \frac{100}{101} \) and \( Q = \frac{2}{1} + \frac{4}{3} + \frac{6}{5} + \ldots + \frac{101}{100} \). Determine exactly the value of \( P + Q \).
1. If \( p^2 = 2020 + q^2 \) and \( p = 10 + q \), compute \( p + q \).

\[
p^2 - q^2 = (p - q)(p + q) = 2020 \quad \text{and} \quad p - q = 10. \therefore 10(p + q) = 2020 \Rightarrow p + q = 202.
\]

2. If \( a \) and \( b \) are positive real numbers and \( \frac{a^2 + b^2}{a^2 + b^2} = 10 \), determine exactly the value of \( \frac{a^3 + b^3}{a^3 + b^3} \).

\[
The \text{ first expression can be rewritten as } \frac{a^2 + b^2}{a^2 + b^2} = 10. \therefore a^2 b^2 = 10 \Rightarrow ab = \sqrt{10}. \text{ The second expression can likewise be written as } \frac{a^3 + b^3}{a^3 + b^3} = (ab)^3 = \left( \sqrt{10} \right)^3 = 10\sqrt{10}.
\]

3. Determine exactly the value for \( x \), given that \( x + 2\sqrt{x} = 2 \).

\[
\text{Graders: } 4 + 2\sqrt{3} \text{ does NOT work.}
\]

\[
\text{Let } y = \sqrt{x}. \text{ Then } y^2 + 2y - 2 = 0 \Rightarrow y = -1 \pm \sqrt{3}. \text{ Choosing the positive value because } y \text{ is the positive square root of } x, x = y^2 = (-1 + \sqrt{3})^2 = 4 - 2\sqrt{3}.
\]

4. Let \( P = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots + \frac{100}{101} \) and \( Q = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \ldots + \frac{101}{100} \). Determine exactly the value of \( P + Q \).

\[
P + Q = \frac{3}{2} + \left( \frac{1}{3} + \frac{1}{2} \right) + \left( \frac{1}{4} + \frac{1}{3} \right) + \ldots + \left( \frac{100}{100} + \frac{100}{100} \right) + \frac{100}{101} = 2 + 2 + 2 + \ldots + 2 + \frac{100}{101} = 2 \cdot 100 + \frac{100}{101} = \frac{20300}{101}.
\]
1. In Figure 1, a quadrilateral is inscribed in a circle. Calculate the sum of the measures of angles $x$ and $y$.

2. In Figure 2, a circle of radius 20 contains three points $A$, $B$, and $C$. Two chords, $AB$ and $AC$, are drawn. If the length of $BC$ is $\frac{5\pi}{3}$, determine exactly the measure of $\angle BAC$.

3. In Figure 3, a solid, formed by two right circular cones, one with a height twice the other but sharing the same base, is inscribed in a sphere of radius 12. Determine exactly the volume of the inscribed solid.

4. Figure 4 shows a cyclic quadrilateral $ABCD$, with perpendicular diagonals $AC$ and $BD$, intersecting at point $G$. From $G$ a segment is drawn perpendicular to $AB$ at $E$ and then extended to intersect $CD$ at $F$. If $BE = 2$, $CG = 6$, and $EG = 4$, determine exactly $CF$.
1. In Figure 1, a quadrilateral is inscribed in a circle. Calculate the sum of the measures of angles $x$ and $y$.

Opposite angles of cyclic quadrilaterals are supplementary. Therefore, $x + 80° + y + 18° = 180° \Rightarrow x + y = 82°$.

2. In Figure 2, a circle of radius 20 contains three points $A$, $B$, and $C$. Two chords, $AB$ and $AC$, are drawn. If the length of $BC$ is $\frac{5\pi}{3}$, determine exactly the measure of $\angle BAC$.

Let $x$ be the central angle intercepting $\overarc{BC}$. Then $\frac{\frac{5\pi}{3}}{40\pi} = \frac{x}{2\pi} \Rightarrow x = \frac{\pi}{12}$. Then $m\angle BAC = \frac{\pi}{12}$.

3. In Figure 3, a solid, formed by two right circular cones, one with a height twice the other but sharing the same base, is inscribed in a sphere of radius 12. Determine exactly the volume of the inscribed solid.

See diagram at right: $3x = 24 \Rightarrow x = 8$. Therefore, $4^2 + r^2 = 12^2 \Rightarrow r^2 = 128$. The volume of the solid is the sum of the volumes of the two cones: $V = \frac{1}{3}\pi (128)8 + \frac{1}{3}\pi (128)16 = 1024\pi$.

4. Figure 4 shows a cyclic quadrilateral $ABCD$, with perpendicular diagonals, $AC$ and $BD$, intersecting at point $G$. From $G$ a segment is drawn perpendicular to $AB$ at $E$ and then extended to intersect $CD$ at $F$. If $BE = 2$, $CG = 6$, and $EG = 4$, determine exactly $CF$.

Since $\angle DCA$ and $\angle DBA$ both intercept $\overarc{DA}$, they are congruent and, therefore, $\triangle CGD \sim \triangle BGA \sim \triangle BEG \sim \triangle GEA$. Also $\angle DCG \equiv \angle AGE \equiv \angle FGC$ and $\angle CDG \equiv \angle BGE \equiv \angle DGF$. So triangles $CFG$ and $GFD$ are isosceles, with $CF = GF = DF = x$. By the Pythagorean Theorem, $BG = 2\sqrt{5}$. The scale factor of triangles $CGD$ and $BEG$ is 3. Therefore, $CD = 6\sqrt{5}$ and $CF = 3\sqrt{5}$.
1. Determine exactly the value of the infinite sum: \(\frac{4}{3} + \frac{4}{9} + \ldots\)

2. What is the value of the sum: \(1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 + \ldots + 242 - 243\) ?

3. If \(f(x) = mx + b\), with \(m < 0\), and \(f(x) = 4 \cdot f^{-1}(x) + 3\) for all \(x\), compute the ordered pair \((m, b)\).

4. When \((\sqrt{3} + \sqrt{2})^9\) is expanded, what is the sum of all the terms that are integers?
NO CALCULATORS are allowed on this event.

1. Determine exactly the value of the infinite sum: \(4 + \frac{4}{3} + \frac{4}{9} + \ldots\)

\[
\frac{4}{1 - \frac{1}{3}} = \frac{12}{3-1} = 6.
\]

2. What is the value of the sum: \(1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 + \ldots + 242 - 243\)?

The plan: Add up \(1 + 2 + 3 + \ldots + 243\) and subtract twice the sum of \(3 + 6 + 9 + \ldots + 243\).

The first sum is \(\left(\frac{1 + 243}{2}\right) \cdot 243 = 122 \cdot 243\).

Twice the second sum is \(2 \cdot 3 \cdot (1 + 2 + 3 + \ldots + 81) =
6 \cdot \left(\frac{1 + 81}{2} \cdot 81\right) = 82 \cdot 243\). So the answer is
\(122 \cdot 243 - 82 \cdot 243 = (122 - 82) \cdot 243 = 40 \cdot 243 = 9720\).

3. If \(f(x) = mx + b\), with \(m < 0\), and \(f(x) = 4 \cdot f^{-1}(x) + 3\) for all \(x\), compute the ordered pair \((m, b)\).

\(f^{-1}(x) = \frac{x - b}{m}\). Therefore, \(mx + b = 4\left(\frac{x - b}{m}\right) + 3 \Rightarrow m^2x + mb = 4x - 4b + 3m\). Since it must be true for all \(x\), let \(x = 0\) and \(x = 1\), producing: \(bm = -4b + 3m\) and \(m^2 + bm = 4 - 4b + 3m\). Substituting yields \(m^2 = 4\), or \(m = \pm 2\). Choosing the negative and solving yields: \(-2b = -4b - 6 \Rightarrow b = -3\).

4. When \((\sqrt{3} + \sqrt{2})^9\) is expanded, what is the sum of all the terms that are integers?

All terms of the expansion are of the form: \(\binom{9}{k}(\sqrt{3})^{9-k}(\sqrt{2})^k = \binom{9}{k}(3)^{\frac{9-k}{2}}(2)^{\frac{k}{2}}\). For this to be an integer, both \(\frac{9-k}{2}\) and \(\frac{k}{3}\) must be integers. This only occurs when \(k = 3\) or \(k = 9\).

Let \(k = 3\): \(\binom{9}{3}(3)^{\frac{9-3}{2}}(2)^{\frac{3}{2}} = 84 \cdot 27 \cdot 2 = 4536\). Let \(k = 9\): \(\binom{9}{9}(3)^{\frac{9-9}{2}}(2)^{\frac{9}{2}} = 1 \cdot 1 \cdot 8 = 8\).
1. What are the coordinates of the vertex of the parabola \( y = 3x^2 - 12x + 7 \) ?

2. A hyperbola has \( y = \frac{5}{2}x + 24 \) and \( y = -\frac{5}{2}x + 4 \) as its asymptotes and has a vertex at \((-4, 19)\). What are the coordinates of the other vertex?

3. For any real number \( m \), the parabola \( f_m(x) = 5x^2 + mx + 4m \) passes through a common point \((a, b)\). Determine exactly the ordered pair \((a, b)\).

4. In Figure 4, two congruent ellipses have perpendicular major axes. Each ellipse passes through the other ellipse’s foci. If the four foci are the vertices of a square with an area of 36, determine exactly the area of one of the ellipses.

**Figure 4**
1. What are the coordinates of the vertex of the parabola \( y = 3x^2 - 12x + 7 \)?

**Completing the square:** \( y = 3(x^2 - 4x + 4) + 7 - 12 = 3(x - 2)^2 - 5 \). So the vertex is \((2, -5)\).

2. A hyperbola has \( y = \frac{5}{2}x + 24 \) and \( y = -\frac{5}{2}x + 4 \) as its asymptotes and has a vertex at \((-4, 19)\). What are the coordinates of the other vertex?

\[
\frac{5}{2}x + 24 = -\frac{5}{2}x + 4 \Rightarrow 5x = -20 \Rightarrow x = -4. \text{ Letting } x = -4 \text{ yields } y = 14. \text{ The given vertex is 5 units above the center } (-4, 14), \text{ so the other vertex is 5 units below the center at } (-4, 9).
\]

3. For any real number \( m \), the parabola \( f_m(x) = 5x^2 + mx + 4m \) passes through a common point \((a, b)\). Determine exactly the ordered pair \((a, b)\).

Since all members of this family of curves pass through \((a,b)\). \( f_j(a) = f_k(a) = b \), for all real \( j \) and \( k \). Therefore, \( 5a^2 + ja + 4j = 5a^2 + ka + 4k \Rightarrow (j - k)a + 4(j - k) = 0 \Rightarrow a = -4 \). Let \( m = 0 \), then \( f_0(-4) = 5(-4)^2 + (-4) \cdot 0 + 4 \cdot 0 = 80 \). Therefore, \((a,b) = (-4, 80)\).

4. In Figure 4, two congruent ellipses have perpendicular major axes. Each ellipse passes through the other ellipse’s foci. If the four foci are the vertices of a square with an area of 36, determine exactly the area of one of the ellipses.

A square whose area is 36 has sides of length 6. Therefore, the diagonal of the square is \(6\sqrt{2}\). This is the length of the minor axis \(2b\) and the distance between the foci \(2c\).

Therefore, in either ellipse, \( b = 3\sqrt{2} \) and \( c = 3\sqrt{2} \). In any ellipse, \( c^2 = a^2 - b^2 \). So \( a = 6 \) and the area, which is \( ab\pi \), is \((6)(3\sqrt{2})\pi = 18\sqrt{2}\pi = 18\pi\sqrt{2} \).
1. Find the constants $A$, $B$, and $C$ such that
\[
\frac{A}{(x-2)(x-4)} + \frac{B}{x-2} + \frac{C}{x(x-4)} = \frac{1}{x}
\]
for all permissible $x$.

2. A triangular bipyramid is formed by six equilateral triangles, all of side length 9. (See Figure 2.) Determine exactly the volume of the largest sphere that can be inscribed inside this triangular bipyramid.

3. Let $a$ and $b$ be positive integers that form the first row of a triangle constructed in the same manner as Pascal’s Triangle. The first three rows of the triangle are shown below. If the sum of all the numbers in the first 10 rows is 6138, find the number of ordered pairs $(a,b)$ that give this sum of 6138.

\[
\begin{array}{ccc}
Row & a & b \\
1 & a & b \\
2 & a & a+b & b \\
3 & a & 2a+b & a+2b & b \\
\end{array}
\]

4. A sequence is defined by: $a_1 = \frac{1}{5}$ and $a_n = 3 - \frac{1}{a_{n-1}}$ for $n \geq 2$. Express $a_6$ as a ratio of two relatively prime integers.

5. If $f(a+b) = f(a) \cdot f(b)$ for all positive integers $a$ and $b$ and $f(1) = i = \sqrt{-1}$, compute $f(2019)$.

6. A circle centered at the origin is tangent to the parabola $y = x^2 - 121$ at two points. If the area of this circle can be expressed as $\frac{a}{b} \pi$, where $a$ and $b$ are relatively prime integers, what is $a+b$?
1. Find the constants $A$, $B$, and $C$ such that
\[ \frac{A}{(x-2)(x-4)} + \frac{B}{x-2} + \frac{C}{x(x-4)} = \frac{1}{x} \]
for all permissible $x$.

2. A triangular bipyramid is formed by six equilateral triangles, all of side length 9. Determine exactly the volume of the largest sphere that can be inscribed inside this triangular bipyramid.

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\[
\begin{array}{ccc}
\text{Row} & a & b \\
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2 & a & a+b & b \\
3 & a & 2a+b & a+2b & b \\
\end{array}
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6. A circle centered at the origin is tangent to the parabola $y = x^2 - 121$ at two points. If the area of this circle can be expressed as $\frac{a}{b}\pi$, where $a$ and $b$ are relatively prime integers, what is $a + b$?
1. Multiplying the equation by \( x(x - 2)(x - 4) \) produces: \( Ax + Bx(x - 4) + C(x - 2) = (x - 2)(x - 4) \). Expanding and simplifying yields: \( Bx^2 + (A - 4B + C)x - 2C = x^2 - 6x + 8 \). Therefore, 
   \[ B = 1, \quad A - 4B + C = -6, \quad \text{and} \quad -2C = 8. \] 
   So \( C = -4 \) and \( A = -6 + 4 + 4 = 2 \).

2. See Figure 2.1. The center of the sphere, point \( O \), will be at the center of \( \Delta ABC \). Let \( M \) be the midpoint of \( BC \). Then \( AM = DM = \frac{2\sqrt{6}}{3} \) and \( OM \) is \( \frac{1}{2} AM = \frac{\sqrt{6}}{2} \). Using the Pythagorean Theorem, \( OD = 3\sqrt{6} \). Now slice the bipyramid in half with plane ADEM, producing the cross-section shown in Figure 2.2. The radius of the sphere will be \( R \), a segment drawn from \( O \), perpendicular to face \( DBC \). \( \Delta DOM \sim \Delta ORM \) and \( \frac{OM}{OM} = \frac{OD}{OD} = \frac{OM}{OM} = \frac{\sqrt{6} \cdot \sqrt{6}}{4} = \sqrt{6} \). 
   So the volume of the sphere is \( V = \frac{4}{3} \pi \left( \sqrt{6} \right)^3 = 8\sqrt{6} \pi \).

<table>
<thead>
<tr>
<th>Row</th>
<th>( a )</th>
<th>( a + b )</th>
<th>( b )</th>
<th>( a + b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( a )</td>
<td>( a + b )</td>
<td>( b )</td>
<td>( 2(a + b) )</td>
</tr>
<tr>
<td>2</td>
<td>( a )</td>
<td>( 2a + b )</td>
<td>( a + b )</td>
<td>( 4(a + b) )</td>
</tr>
<tr>
<td>3</td>
<td>( a )</td>
<td>( 3a + b )</td>
<td>( 3a + 3b )</td>
<td>( 8(a + b) )</td>
</tr>
<tr>
<td>4</td>
<td>( a )</td>
<td>( 4a + b )</td>
<td>( 6a + 4b )</td>
<td>( 16(a + b) )</td>
</tr>
</tbody>
</table>

The sum of these row sums forms a geometric series with a common ratio of 2. Therefore, the sum of the first 10 rows is 
\( (a + b) \left( 1 + 2 + 2^2 + 2^3 + \ldots + 2^9 \right) = (a + b) \left( \frac{1 - 2^{10}}{1 - 2} \right) = 1023(a + b) = 6138. \) So \( (a + b) = 6 \) and there are \( 5 \) possible ordered pairs: \((1, 5), (2, 4), (3, 3), (4, 2), \) and \((5, 1)\).

4. Consider the general case when \( a_1 = x \). \( a_2 = 3 - \frac{1}{x} = \frac{3x - 1}{x} \). \( a_3 = 3 - \frac{x}{3x - 1} = \frac{8x - 3}{3x - 1} \). \( a_4 = 3 - \frac{3x - 1}{8x - 3} = \frac{21x - 8}{8x - 3} \).
   Notice the Fibonacci numbers! In general, \( a_n = \frac{F_{2n}x - F_{2n-2}}{F_{2n-2}x - F_{2n-4}} \). So \( a_6 = \frac{144x - 55}{55x - 21} = \frac{144 \cdot \frac{1}{2} - 55}{55 \cdot \frac{1}{2} - 21} = \frac{131}{50} \).

5. \( f(2) = f(1) \cdot f(1) = -1 \cdot -1 = 1 \). \( f(3) = f(2) \cdot f(1) = f(2) \cdot (-1) = -1 \). \( f(4) = f(2) \cdot f(2) = f(2)^2 = 1 \). Therefore, \( f(5) = f(1) = i \) and the values of \( f(x) \) are cyclic with a period of 4. Since \( 2019 \equiv 3 \mod 4 \), \( f(2019) = -i \).

6. The equation of the circle is \( x^2 + y^2 = r^2 \) and the equation for the parabola is \( y + 121 = x^2 \). Substituting and rearranging yields: \( y^2 + y + 121 - r^2 = 0 \). Since the parabola is symmetric with respect to the \( y \)-axis, the tangency points of the circle and parabola have the same \( y \)-coordinate. Therefore, the quadratic must have only one solution and so the discriminant must be 0. So \( 1 - 4(1)(121 - r^2) = 0 \Rightarrow r^2 = \frac{483}{4} \). The area of the circle is \( \frac{483}{4} \pi \) and \( a + b = 487 \).