1. The letters of $ABACUS$ are replaced with distinct digits 0 through 9 with different letters representing different digits. If the three-digit integer $ABA$ is a perfect cube and the three-digit integer $CUS$ is a perfect square divisible by 12, what is $A + B + A + C + U + S$?

2. One day a store made $4400 by selling 60 items of clothing consisting of pants for $30, jackets for $80, and suits for $120. What is the greatest number of suits the store could have sold that day?

3. My cat Delta meows, hisses, and purrs. I heard her make at least one of these sounds on each of the past 30 days. In these 30 days, she hissed on 8 of the days, purred on 14 of the days, and meowed on 18 of the days. On 3 of the days, I heard her meow and hiss but not purr, and on 3 of the days, I heard her purr and hiss but not meow. On one day, she made all three sounds! On how many of these 30 days did I hear her meow and purr but not hiss?

4. The function $f$ is defined recursively by $f(1) = f(2) = 1$ and $f(n) = f(n - 1) - f(n - 2) + n$ for all integers $n \geq 3$. What is the value of $f(1234)$?
1. The letters of \textit{ABACUS} are replaced with distinct digits 0 through 9 with different letters representing different digits. If the three-digit integer \textit{ABA} is a perfect cube and the three-digit integer \textit{CUS} is a perfect square divisible by 12, what is \(A + B + A + C + U + S\)?

The 3-digit cubes are 125, 216, 343, 512, and 729. So \textit{ABA} is 343. The 3-digit squares divisible by 12 are 144, 324, 576, and 900. So \textit{CUS} is 576. \(A + B + A + C + U + S = 3 + 4 + 3 + 5 + 7 + 6 = 28\).

2. One day a store made $4400 by selling 60 items of clothing consisting of pants for $30, jackets for $80, and suits for $120. What is the greatest number of suits the store could have sold that day?

Let \(x\) = # of pants, \(y\) = # of jackets, and \(z\) = # of suits. Then \(x + y + z = 60\) and \(30x + 80y + 120z = 4400\). Multiplying the first equation by \(-30\) and adding it to the second equation yields \(5y + 9z = 260\) ⇒ \(z = \frac{260 - 5y}{9}\). Since \(z\) must be an integer, the minimum value of \(y\) is 7, so \(z = 25\).

3. My cat Delta meows, hisses, and purrs. I heard her make at least one of these sounds on each of the past 30 days. In these 30 days, she hissed on 8 of the days, purred on 14 of the days, and meowed on 18 of the days. On 3 of the days, I heard her meow and hiss but not purr, and on 3 of the days, I heard her purr and hiss but not meow. On one day, she made all three sounds! On how many of these 30 days did I hear her meow and purr but not hiss?

Place the data in a Venn diagram and \(1 + 3 + 1 + 3 + 14 - x + x + 10 - x = 30 \Rightarrow 32 - x = 30 \Rightarrow x = 2\).

4. The function \(f\) is defined recursively by \(f(1) = f(2) = 1\) and \(f(n) = f(n-1) - f(n-2) + n\) for all integers \(n \geq 3\). What is the value of \(f(1234)\)?

Create a table and notice a pattern of period six:

\[
\begin{array}{cccccc}
 n & f(n) & n & f(n) & n & f(n) \\
 1 & 1 & 7 & 7 & 13 & 13 = n + 0 \\
 2 & 1 & 8 & 7 & 14 & 13 = n - 1 \\
 3 & 3 & 9 & 9 & 15 & 15 = n + 0 \\
 4 & 6 & 10 & 12 & 16 & 18 = n + 2 \\
 5 & 8 & 11 & 14 & 17 & 20 = n + 3 \\
 6 & 8 & 12 & 14 & 18 & 20 = n + 2 \\
\end{array}
\]

Since \(1234 \equiv 4 \text{ mod } 6\), \(f(1234) = n + 2 = 1236\).
1. For a sphere of radius 3, determine exactly the ratio of its volume to its surface area.

2. Let $P$ and $Q$ be similar rectangular prisms. Given that $P$ has a surface area of 100 $m^2$ and a volume of 20 $m^3$, and that $Q$ has a surface area of 900 $m^2$ and a volume of $x$ $m^3$, determine exactly the value of $x$.

3. The diagonals of trapezoid $ABCD$, with bases $AB$ and $DC$, intersect at $P$. Given that $AB = 5$, $DC = 7$, and $[ABCD] = 144$, determine exactly the distance from $P$ to $AB$.

$SH = \underline{\hspace{2cm}}$

4. In square $MATH$ a point $S$ is chosen on side $TH$. Then a circle (as large as possible) of radius $r$ is inscribed in quadrilateral $MASH$, and a circle of radius $s$ is inscribed in triangle $SAT$. Given that $AT = 1$ and $r : s = 5 : 4$, determine exactly the length of $SH$.

Figure 4
1. For a sphere of radius 3, determine exactly the ratio of its volume to its surface area.

\[
\frac{V}{SA} = \frac{\frac{4}{3} \pi r^3}{4 \pi r^2} = \frac{r}{3} = \frac{3}{3} = 1.
\]

2. Let \( P \) and \( Q \) be similar rectangular prisms. Given that \( P \) has a surface area of 100 \( m^2 \) and a volume of 20 \( m^3 \), and that \( Q \) has a surface area of 900 \( m^2 \) and a volume of \( x \) \( m^3 \), determine exactly the value of \( x \).

The ratio of areas is 1:9, so the ratio of side lengths is 1:3. The ratio of volumes must be 1:27. So \( x = 20 \cdot 27 = 540 \).

3. The diagonals of trapezoid \( ABCD \), with bases \( \overline{AB} \) and \( \overline{DC} \), intersect at \( P \). Given that \( \overline{AB} = 5 \), \( \overline{DC} = 7 \), and \( \left[ ABCD \right] = 144 \), determine exactly the distance from \( P \) to \( \overline{AB} \).

Let \( h \) be the height of the trapezoid, then \( \frac{1}{2}(5+7)h = 144 \Rightarrow h = 24 \). Since \( \overline{AB} \parallel \overline{DC} \), \( \triangle APB \sim \triangle CPD \), with a scale factor of 5:7. Let the distance from \( P \) to \( \overline{AB} \) be 5\( x \) and the distance from \( P \) to \( \overline{DC} \) be 7\( x \). Then \( 5x + 7x = 24 \Rightarrow x = 2 \). So the distance from \( P \) to \( \overline{AB} \) is 10.

4. In square \( MATH \) a point \( S \) is chosen on side \( \overline{TH} \). Then a circle (as large as possible) of radius \( r \) is inscribed in quadrilateral \( MASH \), and a circle of radius \( s \) is inscribed in triangle \( SAT \). Given that \( AT = 1 \) and \( r : s = 5 : 4 \), determine exactly the length of \( \overline{SH} \).

Extend \( \overline{MH} \) and \( \overline{AS} \) to meet at \( R \). \( \triangle ARM \) and \( \triangle SAT \) are similar and the given circles are the incircles of these triangles. Therefore, the ratio of similarity is 5:4. Let \( SH = x \) and \( TS = 1-x \).

\[
\frac{MA}{TS} = \frac{5}{4} = \frac{1}{1-x} \Rightarrow 5 - 5x = 4 \Rightarrow x = \frac{1}{5}.
\]
1. How many distinct ways are there to arrange the letters in MSHSML?

2. When tossing two 8-sided dice, the sides of each die being numbered 1 through 8, determine exactly the probability of rolling two numbers \(a\) and \(b\), such that \(|a - b| < 3\).

3. The probability of event \(A\), written as \(P(A)\), equals 0.4, and \(P(A \text{ and } B) = 0.172\). If \(A\) and \(B\) are independent events, determine exactly \(P(A \text{ or } B)\).

4. Determine how many numbers between 1 and 200, inclusive, have exactly four integer factors, including 1 and the number itself.
1. How many distinct ways are there to arrange the letters in MSHSML?

\[
\frac{6!}{2! \cdot 2!} = 180 \text{ ways.}
\]

2. When tossing two 8-sided dice, the sides of each die being numbered 1 through 8, determine exactly the probability of rolling two numbers \(a\) and \(b\), such that \(|a - b| < 3\).

Make a sample space: There are 15 ×'s in the upper right triangle representing pairs that won't work. 64 − 2 · 15 = 34 that work.

3. The probability of event \(A\), written as \(P(A)\), equals 0.4, and \(P(A \text{ and } B) = 0.172\). If \(A\) and \(B\) are independent events, determine exactly \(P(A \text{ or } B)\).

If \(A\) and \(B\) are independent, \(P(A \text{ and } B) = P(A) \cdot P(B)\). Therefore, \(P(B) = \frac{0.172}{0.4} = 0.43\).

\[
P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = 0.4 + 0.43 - 0.172 = 0.658
\]

4. Determine how many numbers between 1 and 200, inclusive, have exactly four integer factors, including 1 and the number itself.

For a number to have four factors a number must be a cube of a prime or a product of two primes. There are 3 of the first type: 8, 27, 125. For the second type, there are 25 primes less than 100:


2 can be paired with 24 larger primes because 2(97) < 200 and 2(101) > 200.
3 can be paired with 16 larger primes because 3(61) < 200 and 3(67) > 200.
5 can be paired with 9 larger primes because 5(37) < 200 and 5(41) > 200.
7 can be paired with 5 larger primes because 7(23) < 200 and 7(29) > 200.
11 can be paired with 2 larger primes because 11(17) < 200 and 11(19) > 200. 13(17) > 200 so there are no more. So there are a total of 59 numbers between 1 and 200 that have exactly 4 factors.
1. A circle has a chord whose length is 12 cm. The distance from the center of the circle to this chord is 6 cm. Determine exactly the area of the circle.

\[ cm^2 \]

2. For \( x \neq \frac{1}{4} \text{ or } \frac{1}{3} \), determine exactly the value of \( x \) such that \( \log_{4x} 9 = \log_{3x} 27 \).

3. Write an inequality representing all values of \( k \) such that \( x^2 + y^2 = 3k \) and \( y = x^2 - k \) intersect at exactly four points.

4. In Figure 4, the area of \( \triangle ABC \) is 480 with \( AC = 20 \) and \( AB = 100 \). \( D \) is the midpoint on \( AB \) and \( E \) is the midpoint on \( AC \). The bisector of \( \angle CAB \) intersects \( ED \) at \( F \) and \( CB \) at \( G \). What is the area of quadrilateral \( FGBD \)?

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Minnesota State High School Mathematics League
2018-19 Meet 5, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

**NO CALCULATORS are allowed on this event.**

Name: ___________________________  Team: ___________________________
1. A circle has a chord whose length is 12 cm. The distance from the center of the circle to this chord is 6 cm. Determine exactly the area of the circle.

\[ 72\pi \text{ cm}^2 \]

2. For \( x \neq \frac{1}{4} \text{ or } \frac{1}{3} \), determine exactly the value of \( x \) such that \( \log_4 x = \log_3 27 \).

\[ \frac{9}{64} \text{ or } 0.140625 \]

3. Write an inequality representing all values of \( k \) such that \( x^2 + y^2 = 3k \) and \( y = x^2 - k \) intersect at exactly four points.

4. In Figure 4, the area of \( \triangle ABC \) is 480 with \( AC = 20 \) and \( AB = 100 \). \( D \) is the midpoint on \( AB \) and \( E \) is the midpoint on \( AC \). The bisector of \( \angle CAB \) intersects \( \overline{ED} \) at \( F \) and \( \overline{CB} \) at \( G \). What is the area of quadrilateral \( FGBD \)?

\[ 300 \text{ cm}^2 \]
1. Use the fact that $2^{24} = 2^{25}$ to find an ordered triple of integers $(a, b, c)$, where $a > b > c > 0$, such that $a^3 + b^4 = c^5$.

2. In Figure 2, a circle is inscribed in $\triangle ABC$ with diameter $PQ$ being perpendicular to $BC$. If $[ABC] = 20$, $[PBC] = 18$, and $BC = 4$, determine exactly the perimeter of $\triangle ABC$, expressed as a quotient of two relatively prime integers.

3. As shown in Figure 3, a chord is drawn in a large circle of radius $R$, dividing it into two regions. A circle is inscribed in the smaller region and two externally tangent circles are inscribed in the larger region. The radii of the three smaller circles each equal 1. Determine exactly the value of $R$, expressed as a quotient of two relatively prime integers.

4. Two teams, A and B, are playing in a tournament. They will play until one team wins four games (no ties allowed). The probability of either team winning the first game is 50%. For both teams the probability of winning the very next game after winning one is 60%, winning a third game after winning two in a row 70%, and winning a fourth game in a row is 75%. Determine exactly the probability team A wins in exactly 5 games.

5. In Figure 5, the large regular hexagon has sides of length 6. The smaller regular hexagon is coincident to two sides of the large hexagon. If the length of the dotted segment is $4\sqrt{3}$, determine exactly the ratio of the area of the small hexagon to the area of the large hexagon.

6. In the complex plane, the solutions to $z^2 = 2 + 2\sqrt{3}i$ and $z^2 = -18 - 18\sqrt{3}i$ form the vertices of a parallelogram. Determine exactly the area of this parallelogram.
1. Use the fact that $2 \cdot 2^{24} = 2^{25}$ to find an ordered triple of integers $(a, b, c)$, where

$a > b > c > 0$, such that $a^3 + b^4 = c^5$.

Graders: This solution is not unique. Check any alternate answers with a calculator.

2. In Figure 2, a circle is inscribed in $\Delta ABC$ with diameter $PQ$ being perpendicular to $BC$. If $[ABC] = 20$, $[PBC] = 18$, and $BC = 4$, determine exactly the perimeter of $\Delta ABC$, expressed as a quotient of two relatively prime integers.

3. As shown in Figure 3, a chord is drawn in a large circle of radius $R$, dividing it into two regions. A circle is inscribed in the smaller region and two externally tangent circles are inscribed in the larger region. The radii of the three smaller circles each equal 1. Determine exactly the value of $R$, expressed as a quotient of two relatively prime integers.

4. Two teams, A and B, are playing in a tournament. They will play until one team wins four games (no ties allowed). The probability of either team winning the first game is 50%. For both teams the probability of winning the very next game after winning one is 60%, winning a third game after winning two in a row 70%, and winning a fourth game in a row is 75%. Determine exactly the probability team A wins in exactly 5 games.

5. In Figure 5, the large regular hexagon has sides of length 6. The smaller regular hexagon is coincident to two sides of the large hexagon. If the length of the dotted segment is $4\sqrt{3}$, determine exactly the ratio of the area of the small hexagon to the area of the large hexagon.

6. In the complex plane, the solutions to $z^7 = 2 + 2\sqrt{3}i$ and $z^7 = -18 - 18\sqrt{3}i$ form the vertices of a parallelogram. Determine exactly the area of this parallelogram.

[2018 AMC 12A, problem #22]
1. \[2^{24} = 2^{25} \Rightarrow 2^{24} + 2^{24} = (2^5)^5 \Rightarrow (2^8)^3 + (2^6)^4 = (2^5)^5 \Rightarrow 256^3 + 64^4 = 32^5.\]

2. Let \( r \) be the radius of the circle. \[PBC = 18 = \frac{1}{2}(4)(2r) \Rightarrow r = \frac{9}{2}.\] \( [ABC] = rs \), where \( r \) is the radius of the inscribed circle and \( s \) is the semi-perimeter. \[s = \frac{[ABC]}{r} = \frac{20}{9} = \frac{40}{9}.\] Therefore, the perimeter \( = 2s = \frac{80}{9}.\)

3. Let \( O \) be the center of the large circle, \( A, B, \) and \( C \) be the centers of the small circles, and \( L, M, \) and \( N \) be three points of tangency as shown in Figure 3.1. Then \( OB = R - 1 \) and \( BN = 1 \), so \( ON^2 = (R - 1)^2 - 1^2 = R^2 - 2R. \) \( OM = OL - LM = R - 2. \) \( ON = MN - OM = 1 - (R - 2) = 3 - R, \) and so \( ON^2 = (3 - R)^2 = 9 - 6R + R^2. \) Therefore, \( R^2 - 2R = R^2 - 6R + 9 \Rightarrow 4R = 9 \Rightarrow R = \frac{9}{4}.\)

4. There are 4 ways for team A to win in 5 games: BAAAA, ABAAA, AABAA, or AAABA. The probabilities of each of these outcomes are:
\[(.5)(.4)(.6)(.7)(.75) + (.5)(.4)(.4)(.6)(.7) + (.5)(.6)(.3)(.4)(.6) + (.5)(.6)(.7)(.25)(.4) = 0.1392.\] (Remember that after team B has won a game, team A has a .4 probability of winning the next game.)

5. See Figure 5.1. Let the side of the small hexagon \( \boxed{BD} \) have length \( x. \) Then \( AB = 2x \) and \( BC = 12 - 2x. \) Since \( \theta = 120^\circ, \) the Law of Cosines gives:
\[(4\sqrt{3})^2 = x^2 + (12 - 2x)^2 - 2(x)(12 - 2x)\cos 120^\circ.\] This simplifies to:
\[x^2 - 12x + 32 = (x - 4)(x - 8) = 0.\] Since \( 8 \) is too large, \( x = 4.\) Since the ratio of the sides is 4:6, the ratio of the areas will be \( \left(\frac{4}{6}\right)^2 = \frac{4}{9}.\)

6. \[z^2 = 2 + 2\sqrt{3}i = 4\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 4\text{cis}(60^\circ).\] Therefore, \( z = \pm 2\text{cis}(30^\circ) = \sqrt{3} + i \) and \( -\sqrt{3} - i. \) Let \( A = (\sqrt{3}, 1) \) and \( B = (-\sqrt{3}, -1). \) Similarly, \( z^2 = -18 - 18\sqrt{3}i = 36\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 36\text{cis}(240^\circ). \) So \( z = \pm 6\text{cis}(120^\circ) = 3 - 3\sqrt{3}i \) and \( -3 + 3\sqrt{3}i. \) Let \( C = (3, -3\sqrt{3}) \) and \( D = (-3, 3\sqrt{3}). \) These points form parallelogram \( ACBD \) and since \( 120^\circ - 30^\circ = 90^\circ, \) the diagonals are perpendicular and so \( ACBD \) is a rhombus. Its area is half the product of the diagonals. Since \( AB = \sqrt{(2\sqrt{3})^2 + 2^2} = 4 \) and \( CD = \sqrt{6^2 + (6\sqrt{3})^2} = 12, \) the area is \( \frac{1}{2} \cdot 4 \cdot 12 = 24.\)